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Engineering Maths First-Aid Kit

Anthony Croft

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A few words from the author ...

Over the past three years I have tried to offer mathematical support to many hundreds of students in the early stages of their degree programmes in engineering.

On many, many occasions I have found that gaps in mathematical knowledge impede progress both in engineering mathematics and also in some of the engineering topics that the students are studying. Sometimes these gaps arise because they have long-since forgotten basic techniques. Sometimes, for a variety of reasons, they seem to have never met certain fundamentals in their previous studies. Whatever the underlying reasons, the only practical remedy is to have available resources which can quickly get to the heart of the problem, which can outline a technique or formula or important results, and, importantly, which students can take away with them. This *Engineering Maths First-Aid Kit* is my attempt at addressing this need.

I am well aware that an approach such as this is not ideal. What many students need is a prolonged and structured course in basic mathematical techniques, when all the foundations can be properly laid and there is time to practice and develop confidence. Piecemeal attempts at helping a student do not really get to the root of the underlying problem. Nevertheless I see this Kit as a realistic and practical damage-limitation exercise, which can provide sufficient sticking plaster to enable the student to continue with the other aspects of their studies which are more important to them.

I have used help leaflets similar to these in the Mathematics Learning Support Centre at Loughborough. They are particularly useful at busy times when I may have just a few minutes to try to help a student, and I would like to revise a topic briefly, and then provide a few simple practice exercises. You should realise that these leaflets are not an attempt to put together a coherent course in engineering mathematics, they are not an attempt to replace a textbook, nor are they intended to be comprehensive in their treatment of individual topics. They are what I say – elements of a First-Aid kit.

I hope that some of your students find that they ease the pain!

Tony Croft
December 1999

Engineering Maths First Aid Kit

1.1

Fractions

Introduction

The ability to work confidently with fractions, both number fractions and algebraic fractions, is an essential skill which underpins all other algebraic processes. In this leaflet we remind you of how number fractions are simplified, added, subtracted, multiplied and divided.

1. Expressing a fraction in its simplest form

In any **fraction** $\frac{p}{q}$, say, the number p at the top is called the **numerator**. The number q at the bottom is called the **denominator**. The number q must never be zero. A fraction can always be expressed in different, yet **equivalent** forms. For example, the two fractions $\frac{2}{6}$ and $\frac{1}{3}$ are equivalent. They represent the same value. A fraction is expressed in its **simplest form** by cancelling any factors which are common to both the numerator and the denominator. You need to remember that factors are numbers which are multiplied together. We note that

$$\frac{2}{6} = \frac{1 \times 2}{2 \times 3}$$

and so there is a factor of 2 which is common to both the numerator and the denominator. This common factor can be cancelled to leave the equivalent fraction $\frac{1}{3}$. Cancelling is equivalent to dividing the top and the bottom by the common factor.

Example

$\frac{12}{20}$ is equivalent to $\frac{3}{5}$ since

$$\frac{12}{20} = \frac{4 \times 3}{4 \times 5} = \frac{3}{5}$$

Exercises

1. Express each of the following fractions in its simplest form:

a) $\frac{12}{16}$, b) $\frac{14}{21}$, c) $\frac{3}{6}$, d) $\frac{100}{45}$, e) $\frac{7}{9}$, f) $\frac{15}{55}$, g) $\frac{3}{24}$.

Answers

1. a) $\frac{3}{4}$, b) $\frac{2}{3}$, c) $\frac{1}{2}$, d) $\frac{20}{9}$, e) $\frac{7}{9}$, f) $\frac{3}{11}$, g) $\frac{1}{8}$.

2. Addition and subtraction of fractions

To add two fractions we first rewrite each fraction so that they both have the same denominator. This denominator is chosen to be the **lowest common denominator**. This is the smallest number which is a multiple of both denominators. Then, the numerators only are added, and the result is divided by the lowest common denominator.

Example
Simplify a) $\frac{7}{16} + \frac{5}{16}$, b) $\frac{7}{16} + \frac{3}{8}$.

Solution
a) In this case the denominators of each fraction are already the same. The lowest common denominator is 16. We perform the addition by simply adding the numerators and dividing the result by the lowest common denominator. So, $\frac{7}{16} + \frac{5}{16} = \frac{7+5}{16} = \frac{12}{16}$. This answer can be expressed in the simpler form $\frac{3}{4}$ by cancelling the common factor 4.

b) To add these fractions we must rewrite them so that they have the same denominator. The lowest common denominator is 16 because this is the smallest number which is a multiple of both denominators. Note that $\frac{3}{8}$ is equivalent to $\frac{6}{16}$ and so we write $\frac{7}{16} + \frac{3}{8} = \frac{7}{16} + \frac{6}{16} = \frac{13}{16}$.

Example
Find $\frac{1}{2} + \frac{2}{3} + \frac{4}{5}$.

Solution
The smallest number which is a multiple of the given denominators is 30. We express each fraction with a denominator of 30.

$$\frac{1}{2} + \frac{2}{3} + \frac{4}{5} = \frac{15}{30} + \frac{20}{30} + \frac{24}{30} = \frac{59}{30}$$

Exercises
1. Evaluate each of the following:

a) $\frac{2}{3} + \frac{5}{4}$, b) $\frac{4}{9} - \frac{1}{2}$, c) $\frac{3}{4} + \frac{5}{6}$, d) $\frac{1}{4} + \frac{1}{3} + \frac{1}{2}$, e) $\frac{2}{5} - \frac{1}{3} - \frac{1}{10}$, f) $\frac{4}{5} + \frac{1}{3} - \frac{3}{4}$.

Answers
1. a) $\frac{23}{12}$, b) $-\frac{1}{18}$, c) $\frac{19}{12}$, d) $\frac{13}{12}$, e) $-\frac{1}{30}$, f) $\frac{23}{60}$.

3. Multiplication and division of fractions

Multiplication of fractions is more straightforward. We simply multiply the numerators to give a new numerator, and multiply the denominators to give a new denominator. For example

$$\frac{5}{7} \times \frac{3}{4} = \frac{5 \times 3}{7 \times 4} = \frac{15}{28}$$

Division is performed by inverting the second fraction and then multiplying. So,

$$\frac{5}{7} \div \frac{3}{4} = \frac{5}{7} \times \frac{4}{3} = \frac{20}{21}$$

Exercises
1. Find a) $\frac{4}{26} \times \frac{13}{7}$, b) $\frac{2}{11} \div \frac{3}{5}$, c) $\frac{2}{1} \times \frac{1}{2}$, d) $\frac{3}{7} \times \frac{2}{5}$, e) $\frac{3}{11} \times \frac{22}{5}$, f) $\frac{5}{6} \div \frac{4}{3}$.

Answers
1. a) $\frac{2}{7}$, b) $\frac{10}{33}$, c) 1, d) $\frac{6}{35}$, e) $\frac{6}{5}$, f) $\frac{5}{8}$.

Engineering Maths First Aid Kit

1.2

Powers and roots

Introduction

Powers are used when we want to multiply a number by itself repeatedly.

1. Powers

When we wish to multiply a number by itself we use **powers**, or **indices** as they are also called.

For example, the quantity $7 \times 7 \times 7 \times 7$ is usually written as 7^4. The number 4 tells us the number of sevens to be multiplied together. In this example, the power, or index, is 4. The number 7 is called the **base**.

Example
$6^2 = 6 \times 6 = 36$. We say that '6 squared is 36', or '6 to the power 2 is 36'.

$2^5 = 2 \times 2 \times 2 \times 2 \times 2 = 32$. We say that '2 to the power 5 is 32'.

Your calculator will be pre-programmed to evaluate powers. Most calculators have a button marked x^y, or simply ˆ. Ensure that you are using your calculator correctly by verifying that $3^{11} = 177147$.

2. Square roots

When 5 is squared we obtain 25. That is $5^2 = 25$.

The reverse of this process is called **finding a square root**. The square root of 25 is 5. This is written as $\sqrt[2]{25} = 5$, or simply $\sqrt{25} = 5$.

Note also that when -5 is squared we again obtain 25, that is $(-5)^2 = 25$. This means that 25 has another square root, -5.

In general, a square root of a number is a number which when squared gives the original number. There are always two square roots of any positive number, one positive and one negative. However, negative numbers do not possess any square roots.

Most calculators have a square root button, probably marked $\sqrt{\ }$. Check that you can use your calculator correctly by verifying that $\sqrt{79} = 8.8882$, to four decimal places. Your calculator will only give the positive square root but you should be aware that the second, negative square root is -8.8882.

An important result is that the square root of a product of two numbers is equal to the product of the square roots of the two numbers. For example

$$\sqrt{16 \times 25} = \sqrt{16} \times \sqrt{25} = 4 \times 5 = 20$$

More generally,

$$\sqrt{ab} = \sqrt{a} \times \sqrt{b}$$

However, your attention is drawn to a common error which students make. It is not true that $\sqrt{a+b} = \sqrt{a} + \sqrt{b}$. Substitute some simple values for yourself to see that this cannot be right.

Exercises

1. Without using a calculator write down the value of $\sqrt{9 \times 36}$.
2. Find the square of the following: a) $\sqrt{2}$, b) $\sqrt{12}$.
3. Show that the square of $5\sqrt{2}$ is 50.

Answers

1. 18 (and also −18). 2. a) 2, b) 12.

3. Cube roots and higher roots

The cube root of a number is the number which when cubed gives the original number. For example, because $4^3 = 64$ we know that the cube root of 64 is 4, written $\sqrt[3]{64} = 4$. All numbers, both positive and negative, possess a single cube root.

Higher roots are defined in a similar way: because $2^5 = 32$, the fifth root of 32 is 2, written $\sqrt[5]{32} = 2$.

Exercises

1. Without using a calculator find a) $\sqrt[3]{27}$, b) $\sqrt[3]{125}$.

Answers

1. a) 3, b) 5.

4. Surds

Expressions involving roots, for example $\sqrt{2}$ and $5\sqrt[3]{2}$, are also known as **surds**. Frequently, in engineering calculations it is quite acceptable to leave an answer in surd form rather than calculating its decimal approximation with a calculator.

It is often possible to write surds in equivalent forms. For example, $\sqrt{48}$ can be written as $\sqrt{3 \times 16}$, that is $\sqrt{3} \times \sqrt{16} = 4\sqrt{3}$.

Exercises

1. Write the following in their simplest surd form: a) $\sqrt{180}$, b) $\sqrt{63}$.
2. By multiplying numerator and denominator by $\sqrt{2}+1$, show that

$$\frac{1}{\sqrt{2}-1} \quad \text{is equivalent to} \quad \sqrt{2}+1$$

Answers

1. a) $6\sqrt{5}$, b) $3\sqrt{7}$.

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1.3

Scientific notation

Introduction

In engineering calculations numbers are often very small or very large, for example 0.00000345 and 870,000,000. To avoid writing lengthy strings of numbers a notation has been developed, known as **scientific notation** which enables us to write numbers much more concisely.

1. Scientific notation

In scientific notation each number is written in the form

$$a \times 10^n$$

where a is a number between 1 and 10 and n is a positive or negative whole number.

Some numbers in scientific notation are

$$5 \times 10^3, \qquad 2.67 \times 10^4, \qquad 7.90 \times 10^{-3}$$

To understand scientific notation you need to be aware that

$$10^1 = 10, \qquad 10^2 = 100, \qquad 10^3 = 1000, \qquad 10^4 = 10000, \qquad \text{and so on,}$$

and also that

$$10^{-1} = \frac{1}{10} = 0.1, \qquad 10^{-2} = \frac{1}{100} = 0.01, \qquad 10^{-3} = \frac{1}{1000} = 0.001, \qquad \text{and so on.}$$

You also need to remember how simple it is to multiply a number by powers of 10. For example, to multiply 3.45 by 10, the decimal point is moved one place to the right to give 34.5. To multiply 29.65 by 100, the decimal point is moved two places to the right to give 2965. In general, to multiply a number by 10^n the decimal point is moved n places to the right if n is a positive whole number and n places to the left if n is a negative whole number. It may be necessary to insert additional zeros to make up the required number of digits.

Example
The following numbers are given in scientific notation. Write them out fully.

a) 5×10^3, b) 2.67×10^4, c) 7.90×10^{-3}.

Solution
a) $5 \times 10^3 = 5 \times 1000 = 5000$.

b) $2.67 \times 10^4 = 26700$.

c) $7.90 \times 10^{-3} = 0.00790$.

Example
Express each of the following numbers in scientific notation.

a) 5670000, b) 0.0098.

Solution
a) $5670000 = 5.67 \times 10^6$.

b) $0.0098 = 9.8 \times 10^{-3}$.

Exercises
1. Express each of the following in scientific notation.

a) 0.00254, b) 82, c) -0.342, d) 1000000.

Answers
1. a) 2.54×10^{-3}, b) 8.2×10, c) -3.42×10^{-1}, d) 1×10^6 or simply 10^6.

2. Using a calculator

Students often have difficulty using a calculator to deal with scientific notation. You may need to refer to your calculator manual to ensure that you are entering numbers correctly. You should also be aware that your calculator can display a number in lots of different forms, including scientific notation. Usually a MODE button is used to select the appropriate format.

Commonly the EXP button is used to enter numbers in scientific notation. (EXP stands for exponent which is another name for a power.) A number like 3.45×10^7 is entered as 3.45EXP 7 and might appear in the calculator window as $3.45\ ^{07}$. Alternatively your calculator may require you to enter the number as 3.45E7 and it may be displayed in the same way. You should seek help if in doubt.

Computer programming languages use similar notation. For example

$$8.25 \times 10^7 \qquad \text{may be programmed as} \qquad 8.25\text{E}7$$

and

$$9.1 \times 10^{-3} \qquad \text{may be programmed as} \qquad 9.1\text{E}{-3}$$

Again, you need to take care and check the required syntax carefully.

A common error is to enter incorrectly numbers which are simply powers of 10. For example, the number 10^7 is erroneously entered as 10E7 which means 10×10^7, that is 10^8. The number 10^7, meaning 1×10^7, should be entered as 1E7.

Check that you are using your calculator correctly by verifying that

$$(3 \times 10^7) \times (2.76 \times 10^{-4}) \times (10^5) = 8.28 \times 10^8$$

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1.4

Factorials

Introduction

In many engineering calculations you will come across the symbol ! which you may not have met before in mathematics classes. This is known as a **factorial**. The factorial is a symbol which is used when we wish to multiply consecutive whole numbers together, as you will see below.

1. Factorials

The number $5 \times 4 \times 3 \times 2 \times 1$ is written as 5!, which is read as 'five factorial'. If you actually perform the multiplication you will find that $5! = 120$. Similarly $7! = 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$ which equals 5040. A rather special case is 0!. This is defined to equal 1 and this might seem somewhat strange. Just learn this!

You will not be required to find factorials of negative numbers or fractions.

Factorials are used in **combination notation** which arises frequently in probability theory. The notation $\binom{n}{r}$ stands for $\frac{n!}{(n-r)!r!}$. For example

$$\binom{6}{4} = \frac{6!}{(6-4)!4!} = \frac{6!}{2!4!}$$

Exercises

1. Without using a calculator evaluate 2!, 3! and 4!.
2. Show that $\frac{5!}{3!}$ equals 20.
3. Explain why $n! = n \times (n-1)!$ for any positive whole number n.
4. Explain why $\dfrac{n!}{(n-1)!} = n$ for any positive whole number n.
5. Evaluate a) $\binom{9}{3}$, b) $\binom{5}{2}$, c) $\binom{6}{1}$.

Answers

1. $2! = 2$ $3! = 6$ and $4! = 24$. Note that $3! = 3 \times 2!$, and that $4! = 4 \times 3!$.

5. a) 84, b) 10, c) 6.

2. Using a calculator to find factorials

Your scientific calculator will be pre-programmed to find factorials. Look for a button marked !, or consult your calculator manual. Check that you can use your calculator to find factorials by verifying that

$$10! = 3628800$$

The modulus of a number

Introduction

In many engineering calculations you will come across the symbol $|\ |$. This is known as the **modulus**.

1. The modulus of a number

The modulus of a number is its absolute size. That is, we disregard any sign it might have.

Example
The modulus of -8 is simply 8.

The modulus of $-\frac{1}{2}$ is $\frac{1}{2}$.

The modulus of 17 is simply 17.

The modulus of 0 is 0.

So, the modulus of a positive number is simply the number.

The modulus of a negative number is found by ignoring the minus sign.

The modulus of a number is denoted by writing vertical lines around the number.

Note also that the modulus of a negative number can be found by multiplying it by -1 since, for example, $-(-8) = 8$.

This observation allows us to define the modulus of a number quite concisely in the following way

$$|x| = \begin{cases} x & \text{if } x \text{ is positive or zero} \\ -x & \text{if } x \text{ is negative} \end{cases}$$

Example

$$|9| = 9, \qquad |-11| = 11, \qquad |0.25| = 0.25, \qquad |-3.7| = 3.7$$

Exercise
1. Draw up a table of values of $|x|$ as x varies between -6 and 6. Plot a graph of $y = |x|$. Compare your graph with the graphs of $y = x$ and $y = -x$.

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2.1

The laws of indices

Introduction

A **power**, or an **index**, is used to write a product of numbers very compactly. The plural of index is **indices**. In this leaflet we remind you of how this is done, and state a number of rules, or laws, which can be used to simplify expressions involving indices.

1. Powers, or indices

We write the expression

$$3 \times 3 \times 3 \times 3 \quad \text{as} \quad 3^4$$

We read this as 'three to the power four'.

Similarly

$$z \times z \times z = z^3$$

We read this as 'z to the power three' or 'z cubed'.

In the expression b^c, the **index** is c and the number b is called the **base**. Your calculator will probably have a button to evaluate powers of numbers. It may be marked x^y. Check this, and then use your calculator to verify that

$$7^4 = 2401 \qquad \text{and} \qquad 25^5 = 9765625$$

Exercises

1. Without using a calculator work out the value of

a) 4^2, b) 5^3, c) 2^5, d) $\left(\frac{1}{2}\right)^2$, e) $\left(\frac{1}{3}\right)^2$, f) $\left(\frac{2}{5}\right)^3$.

2. Write the following expressions more concisely by using an index.

a) $a \times a \times a \times a$, b) $(yz) \times (yz) \times (yz)$, c) $\left(\frac{a}{b}\right) \times \left(\frac{a}{b}\right) \times \left(\frac{a}{b}\right)$.

Answers

1. a) 16, b) 125, c) 32, d) $\frac{1}{4}$, e) $\frac{1}{9}$, f) $\frac{8}{125}$.

2. a) a^4, b) $(yz)^3$, c) $\left(\frac{a}{b}\right)^3$.

2. The laws of indices

To manipulate expressions involving indices we use rules known as the **laws of indices**. The laws should be used precisely as they are stated – do not be tempted to make up variations of your own! The three most important laws are given here:

First law

$$a^m \times a^n = a^{m+n}$$

When expressions with the same base are multiplied, the indices are added.

Example
We can write

$$7^6 \times 7^4 = 7^{6+4} = 7^{10}$$

You could verify this by evaluating both sides separately.

Example

$$z^4 \times z^3 = z^{4+3} = z^7$$

Second law

$$\frac{a^m}{a^n} = a^{m-n}$$

When expressions with the same base are divided, the indices are subtracted.

Example
We can write

$$\frac{8^5}{8^3} = 8^{5-3} = 8^2 \qquad \text{and similarly} \qquad \frac{z^7}{z^4} = z^{7-4} = z^3$$

Third law

$$(a^m)^n = a^{mn}$$

Note that m and n have been multiplied to yield the new index mn.

Example

$$(6^4)^2 = 6^{4\times 2} = 6^8 \qquad \text{and} \qquad (e^x)^y = e^{xy}$$

It will also be useful to note the following important results:

$$a^0 = 1, \qquad a^1 = a$$

Exercises
1. In each case choose an appropriate law to simplify the expression:
a) $5^3 \times 5^{13}$, b) $8^{13} \div 8^5$, c) $x^6 \times x^5$, d) $(a^3)^4$, e) $\frac{y^7}{y^3}$, f) $\frac{x^8}{x^7}$.

2. Use one of the laws to simplify, if possible, $a^6 \times b^5$.

Answers
1. a) 5^{16}, b) 8^8, c) x^{11}, d) a^{12}, e) y^4, f) $x^1 = x$.

2. This cannot be simplified because the bases are not the same.

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2.2

Negative and fractional powers

Introduction

Sometimes it is useful to use negative and fractional powers. These are explained in this leaflet.

1. Negative powers

Sometimes you will meet a number raised to a negative power. This is interpreted as follows:

$$a^{-m} = \frac{1}{a^m}$$

This can be rearranged into the alternative form:

$$a^m = \frac{1}{a^{-m}}$$

Example

$$3^{-2} = \frac{1}{3^2}, \qquad \frac{1}{5^{-2}} = 5^2, \qquad x^{-1} = \frac{1}{x^1} = \frac{1}{x}, \qquad x^{-2} = \frac{1}{x^2}, \qquad 2^{-5} = \frac{1}{2^5} = \frac{1}{32}$$

Exercises

1. Write the following using only positive powers:

a) $\dfrac{1}{x^{-6}}$, b) x^{-12}, c) t^{-3}, d) $\frac{1}{4^{-3}}$, e) 5^{-17}.

2. Without using a calculator evaluate a) 2^{-3}, b) 3^{-2}, c) $\frac{1}{4^{-2}}$, d) $\frac{1}{2^{-5}}$, e) $\frac{1}{4^{-3}}$.

Answers

1. a) x^6, b) $\frac{1}{x^{12}}$, c) $\frac{1}{t^3}$, d) 4^3, e) $\frac{1}{5^{17}}$.

2. a) $2^{-3} = \frac{1}{2^3} = \frac{1}{8}$, b) $\frac{1}{9}$, c) 16, d) 32, e) 64.

2. Fractional powers

To understand fractional powers you first need to have an understanding of roots, and in particular square roots and cube roots. If necessary you should consult leaflet *1.2 Powers and Roots*.

When a number is raised to a fractional power this is interpreted as follows:

$$a^{1/n} = \sqrt[n]{a}$$

So,

$$a^{1/2} \quad \text{is a square root of } a$$
$$a^{1/3} \quad \text{is the cube root of } a$$
$$a^{1/4} \quad \text{is a fourth root of } a$$

Example

$$3^{1/2} = \sqrt[2]{3}, \qquad 27^{1/3} = \sqrt[3]{27} \;\text{ or }\; 3, \qquad 32^{1/5} = \sqrt[5]{32} = 2,$$
$$64^{1/3} = \sqrt[3]{64} = 4, \qquad 81^{1/4} = \sqrt[4]{81} = 3$$

Fractional powers are useful when we need to calculate roots using a scientific calculator. For example to find $\sqrt[7]{38}$ we rewrite this as $38^{1/7}$ which can be evaluated using a scientific calculator. You may need to check your calculator manual to find the precise way of doing this, probably with the buttons x^y or $x^{1/y}$.

Check that you are using your calculator correctly by confirming that

$$38^{1/7} = 1.6814 \quad (4\,\text{dp})$$

More generally $a^{m/n}$ means $\sqrt[n]{a^m}$, or equivalently $(\sqrt[n]{a})^m$.

$$a^{m/n} = \sqrt[n]{a^m} \text{ or equivalently } \left(\sqrt[n]{a}\right)^m$$

Example

$$8^{2/3} = (\sqrt[3]{8})^2 = 2^2 = 4 \qquad \text{and} \qquad 32^{3/5} = (\sqrt[5]{32})^3 = 2^3 = 8$$

Exercises

1. Use a calculator to find a) $\sqrt[5]{96}$, b) $\sqrt[4]{32}$.
2. Without using a calculator, evaluate a) $4^{3/2}$, b) $27^{2/3}$.
3. Use the third law of indices to show that

$$a^{m/n} = \sqrt[n]{a^m}$$

and equivalently

$$a^{m/n} = \left(\sqrt[n]{a}\right)^m$$

Answers

1. a) 2.4915, b) 2.3784. 2. a) $4^{3/2} = 8$, b) $27^{2/3} = 9$.

2.3

Removing brackets 1

Introduction

In order to simplify mathematical expressions it is frequently necessary to 'remove brackets'. This means to rewrite an expression which includes bracketed terms in an equivalent form, but without any brackets. This operation must be carried out according to certain rules which are described in this leaflet.

1. The associativity and commutativity of multiplication

Multiplication is said to be a **commutative** operation. This means, for example, that 4×5 has the same value as 5×4. Either way the result is 20. In symbols, xy is the same as yx, and so we can interchange the order as we wish.

Multiplication is also an **associative** operation. This means that when we want to multiply three numbers together such as $4 \times 3 \times 5$ it doesn't matter whether we evaluate 4×3 first and then multiply by 5, or evaluate 3×5 first and then multiply by 4. That is

$$(4 \times 3) \times 5 \qquad \text{is the same as} \qquad 4 \times (3 \times 5)$$

where we have used brackets to indicate which terms are multiplied first. Either way, the result is the same, 60. In symbols, we have

$$(x \times y) \times z \qquad \text{is the same as} \qquad x \times (y \times z)$$

and since the result is the same either way, the brackets make no difference at all and we can write simply $x \times y \times z$ or simply xyz. When mixing numbers and symbols we usually write the numbers first. So

$$\begin{aligned} 7 \times a \times 2 &= 7 \times 2 \times a \qquad \text{through commutativity} \\ &= 14a \end{aligned}$$

Example
Remove the brackets from a) $4(2x)$, b) $a(5b)$.

Solution
a) $4(2x)$ means $4 \times (2 \times x)$. Because of associativity of multiplication the brackets are unnecessary and we can write $4 \times 2 \times x$ which equals $8x$.

b) $a(5b)$ means $a \times (5b)$. Because of commutativity this is the same as $(5b) \times a$, that is $(5 \times b) \times a$. Because of associativity the brackets are unnecessary and we write simply $5 \times b \times a$ which equals $5ba$. Note that this is also equal to $5ab$ because of commutativity.

Exercises

1. Simplify

a) $9(3y)$, b) $(5x) \times (5y)$, c) $3(-2a)$, d) $-7(-9x)$, e) $12(3m)$, f) $5x(y)$.

Answers

1. a) $27y$, b) $25xy$, c) $-6a$, d) $63x$, e) $36m$, f) $5xy$.

2. Expressions of the form $a(b + c)$ and $a(b - c)$

Study the expression $4 \times (2+3)$. By working out the bracketed term first we obtain 4×5 which equals 20. Note that this is the same as multiplying both the 2 and 3 separately by 4, and then adding the results. That is

$$4 \times (2 + 3) = 4 \times 2 \quad + \quad 4 \times 3 = 8 + 12 = 20$$

Note the way in which the '4' multiplies both the bracketed numbers, '2' and '3'. We say that the '4' distributes itself over both the added terms in the brackets – *multiplication is distributive over addition*.

Now study the expression $6 \times (8 - 3)$. By working out the bracketed term first we obtain 6×5 which equals 30. Note that this is the same as multiplying both the 8 and the 3 by 6 before carrying out the subtraction:

$$6 \times (8 - 3) = 6 \times 8 \quad - \quad 6 \times 3 = 48 - 18 = 30$$

Note the way in which the '6' multiplies both the bracketed numbers. We say that the '6' distributes itself over both the terms in the brackets – *multiplication is distributive over subtraction*. Exactly the same property holds when we deal with symbols.

$$a(b + c) = ab + ac \qquad a(b - c) = ab - ac$$

Example

$4(5 + x)$ is equivalent to $4 \times 5 \; + \; 4 \times x$ which equals $20 + 4x$.

$5(a - b)$ is equivalent to $5 \times a \; - \; 5 \times b$ which equals $5a - 5b$.

$7(x - 2y)$ is equivalent to $7 \times x \; - \; 7 \times 2y$ which equals $7x - 14y$.

$-4(5 + x)$ is equivalent to $-4 \times 5 \; + \; -4 \times x$ which equals $-20 - 4x$.

$-5(a - b)$ is equivalent to $-5 \times a \; - \; -5 \times b$ which equals $-5a + 5b$.

$-(a + b)$ is equivalent to $-a - b$.

Exercises

Remove the brackets from each of the following expressions, simplifying your answers where appropriate.

1. $8(3 + 2y)$. 2. $7(-x + y)$. 3. $-7(-x + y)$. 4. $-(3 + 2x)$. 5. $-(3 - 2x)$.

6. $-(-3 - 2x)$. 7. $x(x + 1)$. 8. $15(x + y)$. 9. $15(xy)$. 10. $11(m + 3n)$.

Answers

1. $24 + 16y$. 2. $-7x + 7y$. 3. $7x - 7y$. 4. $-3 - 2x$. 5. $-3 + 2x$. 6. $3 + 2x$.
7. $x^2 + x$. 8. $15x + 15y$. 9. $15xy$. 10. $11m + 33n$.

Engineering Maths First Aid Kit

2.4

Removing brackets 2

Introduction

In this leaflet we show the correct procedure for writing expressions of the form $(a+b)(c+d)$ in an alternative form without brackets.

1. Expressions of the form $(a+b)(c+d)$

In the expression $(a+b)(c+d)$ it is intended that each term in the first bracket multiplies each term in the second.

$$(a+b)(c+d) = ac + bc + ad + bd$$

Example
Removing the brackets from $(5+a)(2+b)$ gives

$$5 \times 2 \;+\; a \times 2 \;+\; 5 \times b \;+\; a \times b$$

which simplifies to

$$10 + 2a + 5b + ab$$

.

Example
Removing the brackets from $(x+6)(x+2)$ gives

$$x \times x \;+\; 6 \times x \;+\; x \times 2 \;+\; 6 \times 2$$

which equals

$$x^2 + 6x + 2x + 12$$

which simplifies to

$$x^2 + 8x + 12$$

Example
Removing the brackets from $(x+7)(x-3)$ gives

$$x \times x \;+\; 7 \times x \;+\; x \times -3 \;+\; 7 \times -3$$

which equals

$$x^2 + 7x - 3x - 21$$

which simplifies to

$$x^2 + 4x - 21$$

Example
Removing the brackets from $(2x + 3)(x + 4)$ gives

$$2x \times x \; + \; 3 \times x \; + \; 2x \times 4 \; + \; 3 \times 4$$

which equals

$$2x^2 + 3x + 8x + 12$$

which simplifies to

$$2x^2 + 11x + 12$$

Occasionally you will need to square a bracketed expression. This can lead to errors. Study the following example.

Example
Remove the brackets from $(x + 1)^2$.

Solution
You need to be clear that when a quantity is squared it is multiplied by itself. So

$$(x + 1)^2 \qquad \text{means} \qquad (x + 1)(x + 1)$$

Then removing the brackets gives

$$x \times x \; + \; 1 \times x \; + \; x \times 1 \; + \; 1 \times 1$$

which equals

$$x^2 + x + x + 1$$

which simplifies to

$$x^2 + 2x + 1$$

Note that $(x + 1)^2$ is not equal to $x^2 + 1$, and more generally $(x + y)^2$ is not equal to $x^2 + y^2$.

Exercises
Remove the brackets from each of the following expressions, simplifying your answers where appropriate.

1. a) $(x + 2)(x + 3)$, b) $(x - 4)(x + 1)$, c) $(x - 1)^2$, d) $(3x + 1)(2x - 4)$.
2. a) $(2x - 7)(x - 1)$, b) $(x + 5)(3x - 1)$, c) $(2x + 1)^2$, d) $(x - 3)^2$.

Answers
1. a) $x^2 + 5x + 6$, b) $x^2 - 3x - 4$, c) $x^2 - 2x + 1$, d) $6x^2 - 10x - 4$.
2. a) $2x^2 - 9x + 7$, b) $3x^2 + 14x - 5$, c) $4x^2 + 4x + 1$, d) $x^2 - 6x + 9$.

Factorising simple expressions

Introduction

Before studying this material you must be familiar with the process of 'removing brackets' as outlined in leaflets 2.3 & 2.4. This is because factorising can be thought of as reversing the process of removing brackets. When we factorise an expression it is written as a product of two or more terms, and these will normally involve brackets.

1. Products and factors

To obtain the **product** of two numbers they are multiplied together. For example the product of 3 and 4 is 3×4 which equals 12. The numbers which are multiplied together are called factors. We say that 3 and 4 are both factors of 12.

Example
The product of x and y is xy.

The product of $5x$ and $3y$ is $15xy$.

Example
$2x$ and $5y$ are factors of $10xy$ since when we multiply $2x$ by $5y$ we obtain $10xy$.

$(x+1)$ and $(x+2)$ are factors of x^2+3x+2 because when we multiply $(x+1)$ by $(x+2)$ we obtain x^2+3x+2.

3 and $x-5$ are factors of $3x-15$ because

$$3(x-5) = 3x - 15$$

2. Common factors

Sometimes, if we study two expressions to find their factors, we might note that some of the factors are the same. These factors are called **common factors**.

Example
Consider the numbers 18 and 12.

Both 6 and 3 are factors of 18 because $6 \times 3 = 18$.

Both 6 and 2 are factors of 12 because $6 \times 2 = 12$.

So, 18 and 12 share a common factor, namely 6.

In fact 18 and 12 share other common factors. Can you find them?

Example
The number 10 and the expression $15x$ share a common factor of 5.

Note that $10 = 5 \times 2$, and $15x = 5 \times 3x$. Hence 5 is a common factor.

Example
$3a^2$ and $5a$ share a common factor of a since

$3a^2 = 3a \times a$ and $5a = 5 \times a$. Hence a is a common factor.

Example
$8x^2$ and $12x$ share a common factor of $4x$ since

$8x^2 = 4x \times 2x$ and $12x = 3 \times 4x$. Hence $4x$ is a common factor.

3. Factorising

To factorise an expression containing two or more terms it is necessary to look for factors which are common to the different terms. Once found, these common factors are written outside a bracketed term. It is ALWAYS possible to check your answers when you factorise by simply removing the brackets again, so you shouldn't get them wrong.

Example
Factorise $15x + 10$.

Solution
First we look for any factors which are common to both $15x$ and 10. The common factor here is 5. So the original expression can be written

$$15x + 10 = 5(3x) + 5(2)$$

which shows clearly the common factor. This common factor is written outside a bracketed term, the remaining quantities being placed inside the bracket:

$$15x + 10 = 5(3x + 2)$$

and the expression has been factorised. We say that the factors of $15x + 10$ are 5 and $3x + 2$. Your answer can be checked by showing

$$5(3x + 2) = 5(3x) + 5(2) = 15x + 10$$

Exercises
Factorise each of the following:

1. $10x + 5y$. 2. $21 + 7x$. 3. $xy - 8x$. 4. $4x - 8xy$.

Answers
1. $5(2x + y)$. 2. $7(3 + x)$. 3. $x(y - 8)$. 4. $4x(1 - 2y)$.

Engineering Maths First Aid Kit

2.6

Factorising quadratics

Introduction

In this leaflet we explain the procedure for factorising quadratic expressions such as $x^2 + 5x + 6$.

1. Factorising quadratics

You will find that you are expected to be able to factorise expressions such as $x^2 + 5x + 6$.

First of all note that by removing the brackets from

$$(x + 2)(x + 3)$$

we find

$$(x + 2)(x + 3) = x^2 + 2x + 3x + 6 = x^2 + 5x + 6$$

When we **factorise** $x^2 + 5x + 6$ we are looking for the answer $(x + 2)(x + 3)$.

It is often convenient to do this by a process of educated guesswork and trial and error.

Example
Factorise $x^2 + 6x + 5$.

Solution
We would like to write $x^2 + 6x + 5$ in the form

$$(\quad + \quad)(\quad + \quad)$$

First note that we can achieve the x^2 term by placing an x in each bracket:

$$(x \quad + \quad)(x \quad + \quad)$$

The next place to look is the constant term in $x^2 + 6x + 5$, that is, 5. By removing the brackets you will see that this is calculated by multiplying the two numbers in the brackets together. We seek two numbers which multiply together to give 5. Clearly 5 and 1 have this property, although there are others. So

$$x^2 + 6x + 5 = (x + 5)(x + 1)$$

At this stage you should always remove the brackets again to check.

The factors of $x^2 + 6x + 5$ are $(x + 5)$ and $(x + 1)$.

Example
Factorise $x^2 - 6x + 5$.

Solution
Again we try to write the expression in the form

$$x^2 - 6x + 5 = (x \quad + \qquad)(x \quad + \qquad)$$

And again we seek two numbers which multiply to give 5. However, this time 5 and 1 will not do, because using these we would obtain a middle term of $+6x$ as we saw in the last example. Trying -5 and -1 will do the trick.

$$x^2 - 6x + 5 = (x - 5)(x - 1)$$

You see that some thought and perhaps a little experimentation is required.

You will need even more thought and care if the coefficient of x^2, that is the number in front of the x^2, is anything other than 1. Consider the following example.

Example
Factorise $2x^2 + 11x + 12$.

Solution
Always start by trying to obtain the correct x^2 term.

We write

$$2x^2 + 11x + 12 = (2x \quad + \qquad)(x \quad + \qquad)$$

Then study the constant term 12. It has a number of pairs of factors, for example 3 and 4, 6 and 2 and so on. By trial and error you will find that the correct factorisation is

$$2x^2 + 11x + 12 = (2x + 3)(x + 4)$$

but you will only realise this by removing the brackets again.

Exercises
1. Factorise each of the following:

a) x^2+5x+4, b) x^2-5x+4, c) x^2+3x-4, d) x^2-3x-4, e) $2x^2-13x-7$,
f) $2x^2+13x-7$, g) $3x^2-2x-1$, h) $3x^2+2x-1$, i) $6x^2+13x+6$.

Answers
1. a) $(x+1)(x+4)$, b) $(x-1)(x-4)$, c) $(x-1)(x+4)$, d) $(x+1)(x-4)$, e) $(2x+1)(x-7)$,
f) $(2x-1)(x+7)$, g) $(3x+1)(x-1)$, h) $(3x-1)(x+1)$, i) $(3x+2)(2x+3)$.

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2.7

Simplifying fractions

Introduction

Fractions involving symbols occur very frequently in engineering mathematics. It is necessary to be able to simplify these and rewrite them in different but equivalent forms. In this leaflet we revise how these processes are carried out. It will be helpful if you have already seen leaflet *1.1 Fractions*.

1. Expressing a fraction in its simplest form

An algebraic fraction can always be expressed in different, yet **equivalent** forms. A fraction is expressed in its **simplest form** by cancelling any factors which are common to both the numerator and the denominator. You need to remember that factors are multiplied together.

For example, the two fractions

$$\frac{7a}{ab} \qquad \text{and} \qquad \frac{7}{b}$$

are equivalent. Note that there is a common factor of a in the numerator and the denominator of $\frac{7a}{ab}$ which can be cancelled to give $\frac{7}{b}$.

> To express a fraction in its simplest form, any factors which are common to both the numerator and the denominator are cancelled.

Notice that cancelling is equivalent to dividing the top and the bottom by the common factor.

It is also important to note that $\frac{7}{b}$ can be converted back to the equivalent fraction $\frac{7a}{ab}$ by multiplying both the numerator and denominator of $\frac{7}{b}$ by a.

> A fraction is expressed in an equivalent form by multiplying both top and bottom by the same quantity, or dividing top and bottom by the same quantity.

Example
The two fractions

$$\frac{10y^2}{15y^5} \qquad \text{and} \qquad \frac{2}{3y^3}$$

are equivalent. Note that

$$\frac{10y^2}{15y^5} = \frac{2 \times 5 \times y \times y}{3 \times 5 \times y \times y \times y \times y \times y}$$

and so there are common factors of 5 and $y \times y$. These can be cancelled to leave $\dfrac{2}{3y^3}$.

Example
The fractions

$$\frac{(x-1)(x+3)}{(x+3)(x+5)} \qquad \text{and} \qquad \frac{(x-1)}{(x+5)}$$

are equivalent. In the first fraction, the common factor $(x+3)$ can be cancelled.

Example
The fractions

$$\frac{2a(3a-b)}{7a(a+b)} \qquad \text{and} \qquad \frac{2(3a-b)}{7(a+b)}$$

are equivalent. In the first fraction, the common factor a can be cancelled. Nothing else can be cancelled.

Example
In the fraction

$$\frac{a-b}{a+b}$$

there are no common factors which can be cancelled. Neither a nor b is a factor of the numerator. Neither a nor b is a factor of the denominator.

Example
Express $\dfrac{5x}{2x+1}$ as an equivalent fraction with denominator $(2x+1)(x-7)$.

Solution
To achieve the required denominator we must multiply both top and bottom by $(x-7)$. That is

$$\frac{5x}{2x+1} = \frac{(5x)(x-7)}{(2x+1)(x-7)}$$

If we wished, the brackets could now be removed to write the fraction as $\dfrac{5x^2-35x}{2x^2-13x-7}$.

Exercises
1. Express each of the following fractions in its simplest form:

a) $\frac{12xy}{16x}$, b) $\frac{14ab}{21a^2b^2}$, c) $\frac{3x^2y}{6x}$, d) $\frac{3(x+1)}{(x+1)^2}$, e) $\frac{(x+3)(x+1)}{(x+2)(x+3)}$, f) $\frac{100x}{45}$, g) $\frac{a+b}{ab}$.

Answers
1. a) $\frac{3y}{4}$, b) $\frac{2}{3ab}$, c) $\frac{xy}{2}$, d) $\frac{3}{x+1}$, e) $\frac{x+1}{x+2}$, f) $\frac{20x}{9}$, g) cannot be simplified. Whilst both a and b are factors of the denominator, neither a nor b is a factor of the numerator.

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2.8

Addition and subtraction

Introduction

Fractions involving symbols occur very frequently in engineering mathematics. It is necessary to be able to add and subtract them. In this leaflet we revise how these processes are carried out. An understanding of writing fractions in equivalent forms is necessary. (See leaflet *2.7 Simplifying fractions*.)

1. Addition and subtraction of fractions

To add two fractions we must first rewrite each fraction so that they both have the same denominator. The denominator is called the **lowest common denominator**. It is the simplest expression which is a multiple of both of the original denominators. Then, the numerators only are added, and the result is divided by the lowest common denominator.

Example
Express as a single fraction

$$\frac{7}{a} + \frac{9}{b}$$

Solution
Both fractions must be written with the same denominator. To achieve this, note that if the numerator and denominator of the first are both multiplied by b we obtain $\frac{7b}{ab}$. This is equivalent to the original fraction – it is merely written in a different form. If the numerator and denominator of the second are both multiplied by a we obtain $\frac{9a}{ab}$. Then the problem becomes

$$\frac{7b}{ab} + \frac{9a}{ab}$$

In this form, both fractions have the same denominator. The lowest common denominator is ab.

Finally we add the numerators and divide the result by the lowest common denominator:

$$\frac{7b}{ab} + \frac{9a}{ab} = \frac{7b + 9a}{ab}$$

Example
Express as a single fraction

$$\frac{2}{x+3} + \frac{5}{x-1}$$

Solution
Both fractions can be written with the same denominator if both the numerator and denominator of the first are multiplied by $x - 1$ and if both the numerator and denominator of the second are multiplied by $x + 3$. This gives

$$\frac{2}{x+3} + \frac{5}{x-1} = \frac{2(x-1)}{(x+3)(x-1)} + \frac{5(x+3)}{(x+3)(x-1)}$$

Then, adding the numerators gives

$$\frac{2(x-1) + 5(x+3)}{(x+3)(x-1)}$$

which, by simplifying the numerator, gives

$$\frac{7x+13}{(x+3)(x-1)}$$

Example
Find $\dfrac{3}{x+1} + \dfrac{2}{(x+1)^2}$

Solution
The simplest expression which is a multiple of the original denominators is $(x+1)^2$. This is the lowest common denominator. Both fractions must be written with this denominator.

$$\frac{3}{x+1} + \frac{2}{(x+1)^2} = \frac{3(x+1)}{(x+1)^2} + \frac{2}{(x+1)^2}$$

Adding the numerators and simplifying we find

$$\frac{3(x+1)}{(x+1)^2} + \frac{2}{(x+1)^2} = \frac{3x+3+2}{(x+1)^2} = \frac{3x+5}{(x+1)^2}$$

Exercises
1. Express each of the following as a single fraction:

a) $\frac{3}{4} + \frac{1}{x}$, b) $\frac{1}{a} - \frac{2}{5b}$, c) $\frac{2}{x^2} + \frac{1}{x}$, d) $2 + \frac{1}{3x}$.

2. Express as a single fraction:

a) $\frac{2}{x+1} + \frac{3}{x+2}$, b) $\frac{2}{x+3} + \frac{5}{(x+3)^2}$, c) $\frac{3x}{x-1} + \frac{1}{x}$, d) $\frac{1}{x-5} - \frac{3}{x+2}$, e) $\frac{1}{2x+1} - \frac{7}{x+3}$.

Answers
1. a) $\frac{3x+4}{4x}$, b) $\frac{5b-2a}{5ab}$, c) $\frac{2+x}{x^2}$, d) $\frac{6x+1}{3x}$.
2. a) $\frac{5x+7}{(x+1)(x+2)}$, b) $\frac{2x+11}{(x+3)^2}$, c) $\frac{3x^2+x-1}{x(x-1)}$, d) $\frac{17-2x}{(x+2)(x-5)}$, e) $-\frac{13x+4}{(x+3)(2x+1)}$.

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2.9

Multiplication and division

Introduction

Fractions involving symbols occur very frequently in engineering mathematics. It is necessary to be able to multiply and divide them. In this leaflet we revise how these processes are carried out. It will be helpful if you have already seen leaflet *1.1 Fractions*.

1. Multiplication and division of fractions

Multiplication of fractions is straightforward. We simply multiply the numerators to give a new numerator, and multiply the denominators to give a new denominator.

Example
Find

$$\frac{4}{7} \times \frac{a}{b}$$

Solution
Simply multiply the two numerators together, and multiply the two denominators together.

$$\frac{4}{7} \times \frac{a}{b} = \frac{4a}{7b}$$

Example
Find

$$\frac{3ab}{5} \times \frac{7}{6a}$$

Solution

$$\frac{3ab}{5} \times \frac{7}{6a} = \frac{21ab}{30a}$$

which, by cancelling common factors, can be simplified to $\frac{7b}{10}$.

Division is performed by inverting the second fraction and then multiplying.

Example
Find $\frac{3}{2x} \div \frac{6}{5y}$.

Solution

$$\begin{aligned}\frac{3}{2x} \div \frac{6}{5y} &= \frac{3}{2x} \times \frac{5y}{6} \\ &= \frac{15y}{12x} \\ &= \frac{5y}{4x}\end{aligned}$$

Example
Find $\dfrac{3}{x+1} \div \dfrac{x}{(x+1)^2}$.

Solution

$$\begin{aligned}\frac{3}{x+1} \div \frac{x}{(x+1)^2} &= \frac{3}{x+1} \times \frac{(x+1)^2}{x} \\ &= \frac{3(x+1)^2}{x(x+1)} \\ &= \frac{3(x+1)}{x}\end{aligned}$$

Exercises
1. Find a) $\frac{1}{3} \times \frac{x}{2}$, b) $\frac{2}{x+1} \times \frac{x}{x-3}$, c) $-\frac{1}{4} \times \frac{3}{5}$, d) $\left(-\frac{1}{x}\right) \times \left(\frac{2}{5y}\right)$, e) $\frac{x+1}{2(x+3)} \times \frac{8}{x+1}$.
2. Simplify
$$\frac{3}{x+2} \div \frac{x}{2x+4}$$
3. Simplify
$$\frac{x+2}{(x+5)(x+4)} \times \frac{x+5}{x+2}$$
4. Simplify
$$\frac{3}{x} \times \frac{3}{y} \times \frac{1}{z}$$
5. Find $\dfrac{4}{3} \div \dfrac{16}{x}$.

Answers
1. a) $\frac{x}{6}$, b) $\frac{2x}{(x+1)(x-3)}$, c) $-\frac{3}{20}$, d) $-\frac{2}{5xy}$, e) $\frac{4}{x+3}$.

2. $\frac{6}{x}$. 3. $\frac{1}{x+4}$. 4. $\frac{9}{xyz}$. 5. $\frac{x}{12}$.

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2.10

Rearranging formulas 1

Introduction

The ability to rearrange formulas or rewrite them in different ways is an important skill in engineering. This leaflet will explain how to rearrange some simple formulas. Leaflet 2.11 deals with more complicated examples.

1. The subject of a formula

Most engineering students will be familiar with Ohm's law which states that $V = IR$. Here, V is a voltage drop, R is a resistance and I is a current. If the values of R and I are known then the formula $V = IR$ enables us to calculate the value of V. In the form $V = IR$, we say that the **subject** of the formula is V. Usually the subject of a formula is on its own on the left-hand side. You may also be familiar with Ohm's law written in either of the forms

$$I = \frac{V}{R} \qquad \text{and} \qquad R = \frac{V}{I}$$

In the first case I is the subject of the formula whilst in the second case R is the subject. If we know values of V and R we can use $I = \frac{V}{R}$ to find I. On the other hand, if we know values of V and I we can use $R = \frac{V}{I}$ to find R. So you see, it is important to be able to write formulas in different ways, so that we can make a particular variable the subject.

2. Rules for rearranging, or transposing, a formula

You can think of a formula as a pair of balanced scales. The quantity on the left is equal to the quantity on the right. If we add an amount to one side of the scale pans, say the left one, then to keep balance we must add the same amount to the pan on the right. Similarly if we take away an amount from the left, we must take the same amount away from the pan on the right. The same applies to formulas. If we add an amount to one side, we must add the same to the other to keep the formula valid. If we subtract an amount from one side we must subtract the same amount from the other. Furthermore, if we multiply the left by any amount, we must multiply the right by the same amount. If we divide the left by any amount we must divide the right by the same amount. When you are trying to rearrange, or **transpose**, a formula, keep these operations clearly in mind.

To transpose or rearrange a formula you may

- add or subtract the same quantity to or from both sides
- multiply or divide both sides by the same quantity.

Later, we shall see that a further group of operations is allowed, but first get some practice with these Examples and Exercises.

Example
Rearrange the formula $y = x + 8$ in order to make x the subject instead of y.

Solution
To make x the subject we must remove the 8 from the right. So, we subtract 8 from the right, but we remember that we must do the same to the left. So

$$\begin{aligned} \text{if} \qquad y &= x + 8, \qquad \text{subtracting 8 yields} \\ y - 8 &= x + 8 - 8 \\ y - 8 &= x \end{aligned}$$

We have x on its own, although it is on the right. This is no problem since if $y - 8$ equals x, then x equals $y - 8$, that is $x = y - 8$. We have succeeded in making x the subject of the formula.

Example
Rearrange the formula $y = 3x$ to make x the subject.

Solution
The reason why x does not appear on its own is that it is multiplied by 3. If we divide $3x$ by 3 we obtain $\frac{3x}{3} = x$. So, we can obtain x on its own by dividing both sides of the formula by 3.

$$\begin{aligned} y &= 3x \\ \frac{y}{3} &= \frac{3x}{3} \\ &= x \end{aligned}$$

Finally $x = \frac{y}{3}$ and we have succeeded in making x the subject of the formula.

Example
Rearrange $y = 11 + 7x$ to make x the subject.

Solution
Starting from $y = 11 + 7x$ we subtract 11 from each side to give $y - 11 = 7x$. Then, dividing both sides by 7 gives $\frac{y-11}{7} = x$. Finally $x = \frac{y-11}{7}$.

Exercises
1. Transpose each of the following formulas to make x the subject.

a) $y = x - 7$, b) $y = 2x - 7$, c) $y = 2x + 7$, d) $y = 7 - 2x$, e) $y = \frac{x}{5}$.

2. Transpose each of the following formulas to make v the subject.

a) $w = 3v$, b) $w = \frac{1}{3}v$, c) $w = \frac{v}{3}$, d) $w = \frac{2v}{3}$, e) $w = \frac{2}{3}v$.

Answers
1. a) $x = y + 7$, b) $x = \frac{y+7}{2}$, c) $x = \frac{y-7}{2}$, d) $x = \frac{7-y}{2}$, e) $x = 5y$.
2. a) $v = \frac{w}{3}$, b) $v = 3w$, c) same as b), d) $v = \frac{3w}{2}$, e) same as d).

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2.11

Rearranging formulas 2

Introduction

This leaflet develops the work started in leaflet 2.10, and shows how more complicated formulas can be rearranged.

1. Further transposition

Remember that when you are trying to rearrange, or **transpose**, a formula, the following operations are allowed.

- Add or subtract the same quantity to or from both sides.
- Multiply or divide both sides by the same quantity.

A further group of operations is also permissible.

A formula remains balanced if we perform the same operation to both sides of it. For example, we can square both sides, we can square-root both sides. We can find the logarithm of both sides. Study the following examples.

Example
Transpose the formula $p = \sqrt{q}$ to make q the subject.

Solution
Here we need to obtain q on its own. To do this we must find a way of removing the square root sign. This can be achieved by squaring both sides since

$$(\sqrt{q})^2 = q$$

So,

$$\begin{aligned} p &= \sqrt{q} \\ p^2 &= q \qquad \text{by squaring both sides} \end{aligned}$$

Finally, $q = p^2$, and we have succeeded in making q the subject of the formula.

Example
Transpose $p = \sqrt{a + b}$ to make b the subject.

Solution

$$\begin{aligned} p &= \sqrt{a+b} \\ p^2 &= a+b \qquad \text{by squaring both sides} \\ p^2 - a &= b \end{aligned}$$

Finally, $b = p^2 - a$, and we have succeeded in making b the subject of the formula.

Example
Make x the subject of the formula $v = \frac{k}{\sqrt{x}}$.

Solution

$$\begin{aligned} v &= \frac{k}{\sqrt{x}} \\ v^2 &= \frac{k^2}{x} \qquad \text{by squaring both sides} \\ xv^2 &= k^2 \qquad \text{by multiplying both sides by } x \\ x &= \frac{k^2}{v^2} \qquad \text{by dividing both sides by } v^2 \end{aligned}$$

and we have succeeded in making x the subject of the formula.

Example
Transpose the formula $T = 2\pi\sqrt{\frac{\ell}{g}}$ for ℓ.

Solution
This must be carried out carefully, in stages, until we obtain ℓ on its own.

$$\begin{aligned} T &= 2\pi\sqrt{\frac{\ell}{g}} \\ \frac{T}{2\pi} &= \sqrt{\frac{\ell}{g}} \qquad \text{by dividing both sides by } 2\pi \\ \left(\frac{T}{2\pi}\right)^2 &= \frac{\ell}{g} \qquad \text{by squaring both sides} \\ \ell &= g\left(\frac{T}{2\pi}\right)^2 \end{aligned}$$

Exercises
1. Make r the subject of the formula $V = \frac{4}{3}\pi r^3$.

2. Make x the subject of the formula $y = 4 - x^2$.

3. Make s the subject of the formula $v^2 = u^2 + 2as$.

Answers
1. $r = \sqrt[3]{\frac{3V}{4\pi}}$. 2. $x = \pm\sqrt{4-y}$. 3. $s = \frac{v^2-u^2}{2a}$.

Engineering Maths First Aid Kit

2.12

Solving linear equations

Introduction

Equations occur in all branches of engineering. They always involve one or more unknown quantities which we try to find when we **solve** the equation. The simplest equations to deal with are **linear equations**. In this leaflet we describe how these are solved.

1. A linear equation

Linear equations are those which can be written in the form

$$ax + b = 0$$

where x is the unknown value, and a and b are known numbers. The following are all examples of linear equations.

$$3x + 2 = 0, \qquad -5x + 11 = 0, \qquad 3x - 11 = 0$$

The unknown does not have to have the symbol x, other letters can be used.

$$3t - 2 = 0, \qquad 7z + 11 = 0, \qquad 3w = 0$$

are all linear equations.

Sometimes you will come across a linear equation which at first sight might not appear to have the form $ax + b = 0$. The following are all linear equations. If you have some experience of solving linear equations, or of transposing formulas, you will be able to check that they can all be written in the standard form.

$$\frac{x-7}{2} + 11 = 0, \qquad \frac{2}{x} = 8, \qquad 6x - 2 = 9$$

2. Solving a linear equation

To solve a linear equation it will be helpful if you know already how to transpose or rearrange formulas. (See leaflets *2.10 & 2.11 Rearranging formulas* for information about this if necessary.)

When solving a linear equation we try to make the unknown quantity the subject of the equation. To do this we may

- add or subtract the same quantity to or from both sides
- multiply or divide both sides by the same quantity.

Example
Solve the equation $x + 7 = 18$.

Solution
We try to obtain x on its own on the left-hand side.

$$\begin{aligned} x + 7 &= 18 \\ x &= 18 - 7 \qquad \text{by subtracting 7 from both sides} \\ x &= 11 \end{aligned}$$

We have solved the equation and found the solution: $x = 11$. The solution is that value of x which can be substituted into the original equation to make both sides the same. You can, and should, check this. Substituting $x = 11$ in the left-hand side of the equation $x + 7 = 18$ we find $11 + 7$ which equals 18, the same as the right-hand side.

Example
Solve the equation $5x + 11 = 22$.

Solution

$$\begin{aligned} 5x + 11 &= 22 \\ 5x &= 22 - 11 \qquad \text{by subtracting 11 from both sides} \\ x &= \frac{11}{5} \qquad \text{by dividing both sides by 5} \end{aligned}$$

Example
Solve the equation $13x - 2 = 11x + 17$.

Solution

$$\begin{aligned} 13x - 2 &= 11x + 17 \\ 13x - 11x - 2 &= 17 \qquad \text{by subtracting } 11x \text{ from both sides} \\ 2x - 2 &= 17 \\ 2x &= 17 + 2 \qquad \text{by adding 2 to both sides} \\ 2x &= 19 \\ x &= \frac{19}{2} \end{aligned}$$

Exercises
1. Solve the following linear equations.

a) $4x + 8 = 0$, b) $3x - 11 = 2$, c) $8(x + 3) = 64$, d) $7(x - 5) = -56$, e) $3c - 5 = 14c - 27$.

Answers
1. a) $x = -2$, b) $x = \frac{13}{3}$, c) $x = 5$, d) $x = -3$, e) $c = 2$.

Engineering Maths First Aid Kit

2.13

Simultaneous equations

Introduction

On occasions you will come across two or more unknown quantities, and two or more equations relating them. These are called **simultaneous equations** and when asked to solve them you must find values of the unknowns which satisfy all the given equations at the same time. In this leaflet we illustrate one way in which this can be done.

1. The solution of a pair of simultaneous equations

The solution of the pair of simultaneous equations

$$3x + 2y = 36, \qquad \text{and} \qquad 5x + 4y = 64$$

is $x = 8$ and $y = 6$. This is easily verified by substituting these values into the left-hand sides to obtain the values on the right. So $x = 8$, $y = 6$ **satisfy** the simultaneous equations.

2. Solving a pair of simultaneous equations

There are many ways of solving simultaneous equations. Perhaps the simplest way is **elimination.** This is a process which involves removing or eliminating one of the unknowns to leave a single equation which involves the other unknown. The method is best illustrated by example.

Example

Solve the simultaneous equations $\begin{array}{rcll} 3x + 2y & = & 36 & (1) \\ 5x + 4y & = & 64 & (2) \end{array}$.

Solution

Notice that if we multiply both sides of the first equation by 2 we obtain an equivalent equation

$$6x + 4y = 72 \qquad (3)$$

Now, if equation (2) is subtracted from equation (3) the terms involving y will be eliminated:

$$\begin{array}{rcll} 6x + 4y & = & 72 \quad - & (3) \\ 5x + 4y & = & 64 & (2) \\ \hline x + 0y & = & 8 & \end{array}$$

So, $x = 8$ is part of the solution. Taking equation (1) (or if you wish, equation (2)) we substitute this value for x, which will enable us to find y:

$$\begin{aligned} 3(8) + 2y &= 36 \\ 24 + 2y &= 36 \\ 2y &= 36 - 24 \\ 2y &= 12 \\ y &= 6 \end{aligned}$$

Hence the full solution is $x = 8$, $y = 6$.

You will notice that the idea behind this method is to multiply one (or both) equations by a suitable number so that either the number of y's or the number of x's are the same, so that subtraction eliminates that unknown. It may also be possible to eliminate an unknown by addition, as shown in the next example.

Example

Solve the simultaneous equations $\begin{aligned} 5x - 3y &= 26 \qquad (1) \\ 4x + 2y &= 34 \qquad (2) \end{aligned}$.

Solution

There are many ways that the elimination can be carried out. Suppose we choose to eliminate y. The number of y's in both equations can be made the same by multiplying equation (1) by 2 and equation (2) by 3. This gives

$$\begin{aligned} 10x - 6y &= 52 \qquad (3) \\ 12x + 6y &= 102 \qquad (4) \end{aligned}$$

If these equations are now added we find

$$\begin{aligned} 10x - 6y &= 52 \ + \qquad (3) \\ 12x + 6y &= 102 \qquad (4) \\ \hline 22x + 0y &= 154 \end{aligned}$$

so that $x = \frac{154}{22} = 7$. Substituting this value for x in equation (1) gives

$$\begin{aligned} 5(7) - 3y &= 26 \\ 35 - 3y &= 26 \\ -3y &= 26 - 35 \\ -3y &= -9 \\ y &= 3 \end{aligned}$$

Hence the full solution is $x = 7$, $y = 3$.

Exercises

Solve the following pairs of simultaneous equations:

a) $\begin{aligned} 7x + y &= 25 \\ 5x - y &= 11 \end{aligned}$, b) $\begin{aligned} 8x + 9y &= 3 \\ x + y &= 0 \end{aligned}$, c) $\begin{aligned} 2x + 13y &= 36 \\ 13x + 2y &= 69 \end{aligned}$, d) $\begin{aligned} 7x - y &= 15 \\ 3x - 2y &= 19 \end{aligned}$.

Answers

a) $x = 3$, $y = 4$, b) $x = -3$, $y = 3$, c) $x = 5$, $y = 2$, d) $x = 1$, $y = -8$.

Quadratic equations 1

Introduction

This leaflet will explain how many quadratic equations can be solved by **factorisation**.

1. Quadratic equations

A **quadratic equation** is an equation of the form $ax^2 + bx + c = 0$, where a, b and c are constants. For example, $3x^2 + 2x - 9 = 0$ is a quadratic equation with $a = 3$, $b = 2$ and $c = -9$.

The constants b and c can have any value including 0. The constant a can have any value except 0. This is to ensure that the equation has an x^2 term. We often refer to a as the coefficient of x^2, to b as the coefficient of x and to c as the constant term. Usually, a, b and c are known numbers, whilst x represents an unknown quantity which we will be trying to find.

2. The solutions of a quadratic equation

To **solve** a quadratic equation we must find values for x which when substituted into the equation make the left-hand and right-hand sides equal. These values are also called **roots**. For example, the value $x = 4$ is a solution of the equation $x^2 - 3x - 4 = 0$ because substituting 4 for x we find

$$4^2 - 3(4) - 4 = 16 - 12 - 4$$

which simplifies to zero, the same as the right-hand side of the equation. There are several techniques which can be used to solve quadratic equations. One of these, *factorisation*, is discussed in this leaflet. You should be aware that not all quadratic equations can be solved by this method. An alternative method which uses a formula is described in leaflet 2.15.

3. Solving a quadratic equation by factorisation

Sometimes, but not always, it is possible to solve a quadratic equation using factorisation. If you need to revise factorisation you should see leaflet *2.6 Factorising quadratics*.

Example
Solve the equation $x^2 + 7x + 12 = 0$ by factorisation.

Solution
We first factorise $x^2 + 7x + 12$ as $(x + 3)(x + 4)$. Then the equation becomes $(x + 3)(x + 4) = 0$.

It is important that you realise that if the product of two quantities is zero, then one or both of the quantities must be zero. It follows that either

$$x + 3 = 0, \text{ that is } x = -3 \qquad \text{or} \qquad x + 4 = 0, \text{ that is } x = -4$$

The roots of $x^2 + 7x + 12 = 0$ are $x = -3$ and $x = -4$.

Example
Solve the quadratic equation $x^2 + 4x - 21 = 0$.

Solution
$x^2 + 4x - 21$ can be factorised as $(x + 7)(x - 3)$. Then

$$\begin{aligned} x^2 + 4x - 21 &= 0 \\ (x + 7)(x - 3) &= 0 \end{aligned}$$

Then either

$$x + 7 = 0, \text{ that is } x = -7 \qquad \text{or} \qquad x - 3 = 0, \text{ that is } x = 3$$

The roots of $x^2 + 4x - 21 = 0$ are $x = -7$ and $x = 3$.

Example
Find the roots of the quadratic equation $x^2 - 10x + 25 = 0$.

Solution

$$x^2 - 10x + 25 = (x - 5)(x - 5) = (x - 5)^2$$

Then

$$\begin{aligned} x^2 - 10x + 25 &= 0 \\ (x - 5)^2 &= 0 \\ x &= 5 \end{aligned}$$

There is one root, $x = 5$. Such a root is called a **repeated root**.

Example
Solve the quadratic equation $2x^2 + 3x - 2 = 0$.

Solution
The equation is factorised to give

$$(2x - 1)(x + 2) = 0$$

so, from $2x - 1 = 0$ we find $2x = 1$, that is $x = \frac{1}{2}$. From $x + 2 = 0$ we find $x = -2$. The two solutions are therefore $x = \frac{1}{2}$ and $x = -2$.

Exercises
1. Solve the following quadratic equations by factorisation.

a) $x^2 + 7x + 6 = 0$, b) $x^2 - 8x + 15 = 0$, c) $x^2 - 9x + 14 = 0$,

d) $2x^2 - 5x - 3 = 0$, e) $6x^2 - 11x - 10 = 0$, f) $6x^2 + 13x + 6 = 0$.

Answers
a) $-1, -6$, b) $3, 5$, c) $2, 7$, d) $3, -\frac{1}{2}$, e) $\frac{5}{2}, -\frac{2}{3}$, f) $x = -\frac{3}{2}, x = -\frac{2}{3}$.

Engineering Maths First Aid Kit

2.15

Quadratic equations 2

Introduction

This leaflet will explain how quadratic equations can be solved using a formula.

1. Solving a quadratic equation using a formula

Any quadratic equation can be solved using the **quadratic formula**.

If

$$ax^2 + bx + c = 0$$

then

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

A quadratic equation has two solutions; one obtained using the positive square root in the formula, and the other obtained using the negative square root. The answers are often referred to as **roots** of the equation.

Example.

Solve the quadratic equation

$$3x^2 + 9x + 4 = 0$$

Solution

Here $a = 3$, $b = 9$ and $c = 4$. Putting these values into the quadratic formula gives

$$\begin{aligned} x &= \frac{-9 \pm \sqrt{9^2 - 4(3)(4)}}{2(3)} \\ &= \frac{-9 \pm \sqrt{81 - 48}}{6} \\ &= \frac{-9 \pm \sqrt{33}}{6} \\ &= \frac{-9 - \sqrt{33}}{6}, \frac{-9 + \sqrt{33}}{6} \\ &= -2.4574, \ -0.5426 \qquad \text{(4dp)} \end{aligned}$$

The roots of $3x^2 + 9x + 4 = 0$ are $x = -2.4574$ and $x = -0.5426$.

Example
Solve the equation $8x^2 + 3x - 4 = 0$.

Solution
Care is needed here because the value of c is negative, that is $c = -4$.

$$\begin{aligned} x &= \frac{-3 \pm \sqrt{3^2 - 4(8)(-4)}}{(2)(8)} \\ &= \frac{-3 \pm \sqrt{137}}{16} \\ &= 0.5440, -0.9190 \qquad \text{(4dp)} \end{aligned}$$

Example
Find the roots of the quadratic equation $9x^2 + 6x + 1 = 0$.

Solution
Here $a = 9$, $b = 6$ and $c = 1$. Using the quadratic formula we have

$$\begin{aligned} x &= \frac{-6 \pm \sqrt{6^2 - 4(9)(1)}}{2(9)} \\ &= \frac{-6 \pm \sqrt{36 - 36}}{18} \\ &= \frac{-6 \pm \sqrt{0}}{18} \\ &= -\frac{6}{18} \\ &= -\frac{1}{3} \end{aligned}$$

In this example there is only one root: $x = -\frac{1}{3}$.

The quantity $b^2 - 4ac$ is called the **discriminant** of the equation. When the discriminant is 0, as in the previous Example, the equation has only one root. If the discriminant is negative we are faced with the problem of finding the square root of a negative number. Such equations require special treatment using what are called *complex numbers*.

Exercises
1. Find the roots of the following quadratic equations:

a) $x^2 + 6x - 8 = 0$, b) $2x^2 - 8x - 3 = 0$, c) $-3x^2 + x + 1 = 0$.

Answers
a) $x = -3 \pm \sqrt{17} = 1.123, -7.123$ (3dp),

b) $x = 2 \pm \frac{\sqrt{22}}{2} = 4.345, -0.345$ (3dp),

c) $x = \frac{1}{6} \pm \frac{\sqrt{13}}{6} = 0.768, -0.434$ (3dp).

Engineering Maths First Aid Kit

2.16

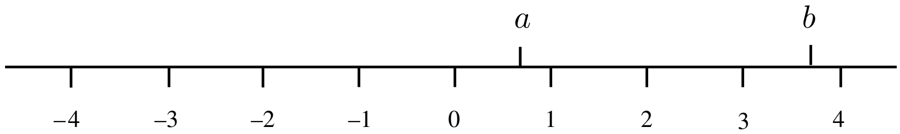
Inequalities

Introduction

The inequality symbols $<$ and $>$ arise frequently in engineering mathematics. This leaflet revises their meaning and shows how expressions involving them are manipulated.

1. The number line and inequality symbols

A useful way of picturing numbers is to use a **number line**. The figure shows part of this line. Positive numbers are on the right-hand side of this line; negative numbers are on the left.



Numbers can be represented on a number line. If $a < b$ then equivalently, $b > a$.

The symbol $>$ means 'greater than'; for example, since 6 is greater than 4 we can write $6 > 4$. Given any number, all numbers to the right of it on the line are greater than the given number. The symbol $<$ means 'less than'; for example, because -3 is less than 19 we can write $-3 < 19$. Given any number, all numbers to the left of it on the line are less than the given number.

For any numbers a and b, note that if a is less than b, then b is greater than a. So the following two statements are equivalent: $a < b$ and $b > a$. So, for example, we can write $4 < 17$ in the equivalent form $17 > 4$.

If $a < b$ and $b < c$ we can write this concisely as $a < b < c$. Similarly if a and b are both positive, with b greater than a we can write $0 < a < b$.

2. Rules for manipulating inequalities

To change or rearrange statements involving inequalities the following rules should be followed:

Rule 1. Adding or subtracting the same quantity from both sides of an inequality leaves the inequality symbol unchanged.

Rule 2. Multiplying or dividing both sides by a **positive** number leaves the inequality symbol unchanged.

Rule 3. Multiplying or dividing both sides by a **negative** number **reverses the inequality**. This means $<$ changes to $>$, and vice versa.

So,

$$\text{if } a < b \quad \text{then} \quad a + c < b + c \qquad \text{using Rule 1}$$

For example, given that $5 < 7$, we could add 3 to both sides to obtain $8 < 10$ which is still true.

Also, using Rule 2,

$$\text{if} \quad a < b \quad \text{and} \quad k \quad \text{is positive, then} \quad ka < kb$$

For example, given that $5 < 8$ we can multiply both sides by 6 to obtain $30 < 48$ which is still true.

Using Rule 3

$$\text{if } a < b \text{ and } k \text{ is negative, then} \quad ka > kb$$

For example, given $5 < 8$ we can multiply both sides by -6 and reverse the inequality to obtain $-30 > -48$, which is a true statement. A common mistake is to forget to reverse the inequality when multiplying or dividing by negative numbers.

3. Solving inequalities

An inequality will often contain an unknown variable, x, say. To **solve** means to find all values of x for which the inequality is true. Usually the answer will be a range of values of x.

Example
Solve the inequality $7x - 2 > 0$.

Solution
We make use of the Rules to obtain x on its own. Adding 2 to both sides gives

$$7x > 2$$

Dividing both sides by the positive number 7 gives

$$x > \frac{2}{7}$$

Hence all values of x greater than $\frac{2}{7}$ satisfy $7x - 2 > 0$.

Example
Find the range of values of x satisfying $x - 3 < 2x + 5$.

Solution
There are many ways of arriving at the correct answer. For example, adding 3 to both sides:

$$x < 2x + 8$$

Subtracting $2x$ from both sides gives

$$-x < 8$$

Multiplying both sides by -1 and **reversing the inequality** gives $x > -8$. Hence all values of x greater than -8 satisfy $x - 3 < 2x + 5$.

Exercises
In each case solve the given inequality.

1. $2x > 9$, 2. $x + 5 > 13$, 3. $-3x < 4$, 4. $7x + 11 > 2x + 5$, 5. $2(x + 3) < x + 1$

Answers
1. $x > 9/2$, 2. $x > 8$, 3. $x > -4/3$, 4. $x > -6/5$, 5. $x < -5$.

Engineering Maths First Aid Kit

2.17

The modulus symbol

Introduction

Inequalities often arise in connection with the modulus symbol. This leaflet describes how.

1. The modulus symbol

The modulus symbol is sometimes used in conjunction with inequalities. For example, $|x| < 1$ means all numbers whose actual size, irrespective of sign, is less than 1. This means any value between -1 and 1. Thus

$$|x| < 1 \quad \text{means} \quad -1 < x < 1$$

Similarly, $|y| > 2$ means all numbers whose actual size, irrespective of sign, is greater than 2. This means any value greater than 2 and any value less than -2. Thus

$$|y| > 2 \quad \text{means} \quad y > 2 \text{ or } y < -2$$

Example
Solve the inequality $|2x + 1| < 3$.

Solution
This is equivalent to $-3 < 2x + 1 < 3$. We treat both parts of the inequality separately.

First consider

$$-3 < 2x + 1$$

Solving this yields $x > -2$.

Now consider the second part, $2x + 1 < 3$. Solving this yields $x < 1$.

Putting both results together we see that $-2 < x < 1$ is the required solution.

Exercises
In each case solve the given inequality.

1. $|3x| < 1$. 2. $|12y + 2| > 5$. 3. $|1 - y| < 3$.

Answers
1. $-\frac{1}{3} < x < \frac{1}{3}$. 2. $y > \frac{1}{4}$ and $y < -\frac{7}{12}$. 3. $-2 < y < 4$.

Graphical solution of inequalities

Introduction

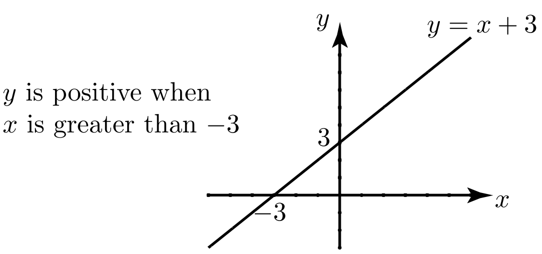
Graphs can be used to solve inequalities. This leaflet illustrates how.

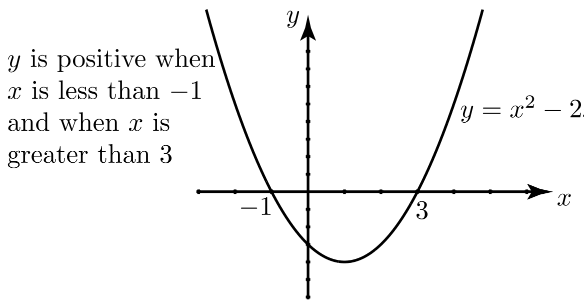
1. Solving inequalities

We start with a very simple example which could be solved very easily using an algebraic method.

Example
Solve the inequality $x + 3 > 0$.

Solution
We seek values of x which make $x + 3$ positive. There are many such values, e.g. try $x = 7$ or $x = -2$. To find all values first let $y = x + 3$. Then the graph of $y = x + 3$ is sketched as shown below. From the graph we see that the y coordinate of any point on the line is positive whenever x has a value greater than -3. That is, $y > 0$ when $x > -3$. But $y = x + 3$, so we can conclude that $x + 3$ will be positive when $x > -3$. We have used the graph to solve the inequality.





Example
Solve the inequality $x^2 - 2x - 3 > 0$.

Solution
We seek values of x which make $x^2 - 2x - 3$ positive. We can find these by sketching a graph of $y = x^2 - 2x - 3$. To help with the sketch, note that by factorising we can write y as $(x+1)(x-3)$. The graph will cross the horizontal axis when $x = -1$ and when $x = 3$. The graph is shown

above on the right. From the graph note that the y coordinate of a point on the graph is positive when either x is greater than 3 or when x is less than -1. That is, $y > 0$ when $x > 3$ or $x < -1$ and so:

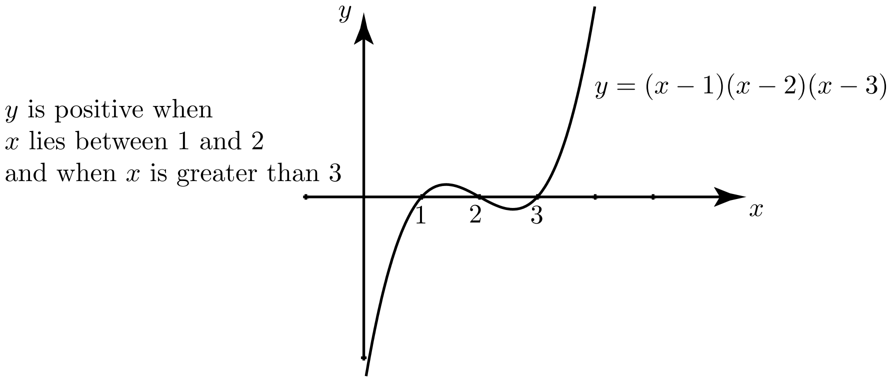
$$x^2 - 2x - 3 > 0 \qquad \text{when} \qquad x > 3 \quad \text{or} \quad x < -1$$

Example

Solve the inequality $(x-1)(x-2)(x-3) > 0$.

Solution

We consider the graph of $y = (x-1)(x-2)(x-3)$ which is shown below. It is evident from the graph that y is positive when x lies between 1 and 2 and also when x is greater than 3. The solution of the inequality is therefore $1 < x < 2$ and $x > 3$.



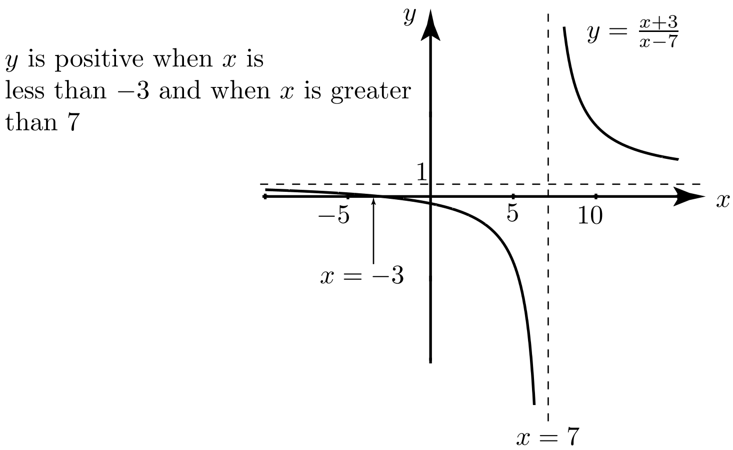
Example

For what values of x is $\frac{x+3}{x-7}$ positive?

Solution

The graph of $y = \frac{x+3}{x-7}$ is shown below. We can see that the y coordinate of a point on the graph is positive when $x < -3$ or when $x > 7$.

$$\frac{x+3}{x-7} > 0 \qquad \text{when} \qquad x < -3 \quad \text{or when} \quad x > 7$$



For drawing graphs like this one a graphical calculator is useful.

What is a logarithm?

Introduction

We use logarithms to write expressions involving powers in a different form. If you can work confidently with powers you should have no problems handling logarithms.

1. Logarithms

Consider the statement

$$100 = 10^2$$

In this statement we say that 10 is the **base** and 2 is the **power** or **index**.

Logarithms are simply an alternative way of writing a statement such as this. We rewrite it as

$$\log_{10} 100 = 2$$

This is read as 'log to the base 10 of 100 is 2'.

As another example, since

$$2^5 = 32$$

we can write

$$\log_2 32 = 5$$

More generally,

$$\text{if} \qquad a = b^c, \qquad \text{then} \qquad \log_b a = c$$

The only restriction that is placed on the value of the base is that it is a positive real number excluding the number 1. In practice logarithms are calculated using only a few common bases. Most frequently you will meet bases 10 and e. The letter e stands for the number 2.718... and is used because it is found to occur in the mathematical description of many physical phenomena. Your calculator will be able to calculate logarithms to bases 10 and e. Usually the 'log' button is used for base 10, and the 'ln' button is used for base e. ('ln' stands for 'natural logarithm'.) Check that you can use your calculator correctly by verifying that

$$\log_{10} 73 = 1.8633 \qquad \text{and} \qquad \log_{\mathrm{e}} 5.64 = 1.7299$$

You may also like to verify the alternative forms

$$10^{1.8633} = 73 \qquad \text{and } \mathrm{e}^{1.7299} = 5.64$$

Occasionally we need to find logarithms to other bases. For example, logarithms to the base 2 are used in communications engineering and information technology. Your calculator can still be used but we need to apply a formula for changing the base. This is dealt with in the leaflet *2.21 Bases other than 10 and e.*

The laws of logarithms

Introduction

There are a number of rules known as the **laws of logarithms**. These allow expressions involving logarithms to be rewritten in a variety of different ways. The laws apply to logarithms of any base but the same base must be used throughout a calculation.

1. The laws of logarithms

The three main laws are stated here:

First Law

$$\log A + \log B = \log AB$$

This law tells us how to add two logarithms together. Adding $\log A$ and $\log B$ results in the logarithm of the product of A and B, that is $\log AB$.

For example, we can write

$$\log_{10} 5 + \log_{10} 4 = \log_{10}(5 \times 4) = \log_{10} 20$$

The same base, in this case 10, is used throughout the calculation. You should verify this by evaluating both sides separately on your calculator.

Second Law

$$\log A - \log B = \log \frac{A}{B}$$

So, subtracting $\log B$ from $\log A$ results in $\log \frac{A}{B}$.

For example, we can write

$$\log_{\mathrm{e}} 12 - \log_{\mathrm{e}} 2 = \log_{\mathrm{e}} \frac{12}{2} = \log_{\mathrm{e}} 6$$

The same base, in this case e, is used throughout the calculation. You should verify this by evaluating both sides separately on your calculator.

Third Law

$$\log A^n = n \log A$$

So, for example

$$\log_{10} 5^3 = 3 \log_{10} 5$$

You should verify this by evaluating both sides separately on your calculator.

Two other important results are

$$\log 1 = 0, \qquad \log_m m = 1$$

The logarithm of 1 to any base is always 0, and the logarithm of a number to the same base is always 1. In particular,

$$\log_{10} 10 = 1, \qquad \text{and} \qquad \log_e e = 1$$

Exercises

1. Use the first law to simplify the following.

a) $\log_{10} 6 + \log_{10} 3$,
b) $\log x + \log y$,
c) $\log 4x + \log x$,
d) $\log a + \log b^2 + \log c^3$.

2. Use the second law to simplify the following.

a) $\log_{10} 6 - \log_{10} 3$,
b) $\log x - \log y$,
c) $\log 4x - \log x$.

3. Use the third law to write each of the following in an alternative form.

a) $3 \log_{10} 5$,
b) $2 \log x$,
c) $\log(4x)^2$,
d) $5 \ln x^4$,
e) $\ln 1000$.

4. Simplify $3 \log x - \log x^2$.

Answers

1. a) $\log_{10} 18$, b) $\log xy$, c) $\log 4x^2$, d) $\log ab^2c^3$.

2. a) $\log_{10} 2$, b) $\log \frac{x}{y}$, c) $\log 4$.

3. a) $\log_{10} 5^3$ or $\log_{10} 125$, b) $\log x^2$, c) $2 \log(4x)$, d) $20 \ln x$ or $\ln x^{20}$,
e) $1000 = 10^3$ so $\ln 1000 = 3 \ln 10$.

4. $\log x$.

Engineering Maths First Aid Kit

2.21

Bases other than 10 and e

Introduction

Occasionally you may need to find logarithms to bases other than 10 and e. For example, logarithms to the base 2 are used in communications engineering and information technology. Your calculator can still be used but we need to apply a formula for changing the base. This leaflet gives this formula and shows how to use it.

1. A formula for change of base

Suppose we want to calculate a logarithm to base 2. The formula states

$$\log_2 x = \frac{\log_{10} x}{\log_{10} 2}$$

So we can calculate base 2 logarithms using base 10 logarithms obtained using a calculator. For example

$$\log_2 36 = \frac{\log_{10} 36}{\log_{10} 2} = \frac{1.556303}{0.301030} = 5.170 \text{ (3dp)}$$

Check this for yourself.

More generally, for any bases a and b,

$$\log_a x = \frac{\log_b x}{\log_b a}$$

In particular, by choosing $b = 10$ we find

$$\log_a x = \frac{\log_{10} x}{\log_{10} a}$$

Use this formula to check that $\log_{20} 100 = 1.5372$.

Exercises

1. Find a) $\log_2 15$, b) $\log_2 56.25$, c) $\log_3 16$.

Answers

1. a) 3.907 (3dp), b) 5.814 (3dp), c) 2.524 (3dp).

Engineering Maths First Aid Kit

2.22

Sigma notation

Introduction

Sigma notation, $\sum$, provides a concise and convenient way of writing long sums. This leaflet explains how.

1. Sigma notation

The sum

$$1 + 2 + 3 + 4 + 5 + \ldots + 10 + 11 + 12$$

can be written very concisely using the capital Greek letter $\sum$ as

$$\sum_{k=1}^{k=12} k$$

The $\sum$ stands for a sum, in this case the sum of all the values of k as k ranges through all whole numbers from 1 to 12. Note that the lower-most and upper-most values of k are written at the bottom and top of the sigma sign respectively. You may also see this written as $\sum_{k=1}^{k=12} k$, or even as $\sum_{k=1}^{12} k$.

Example
Write out explicitly what is meant by

$$\sum_{k=1}^{k=5} k^3$$

Solution
We must let k range from 1 to 5, cube each value of k, and add the results:

$$\sum_{k=1}^{k=5} k^3 = 1^3 + 2^3 + 3^3 + 4^3 + 5^3$$

Example
Express $\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}$ concisely using sigma notation.

Solution
Each term takes the form $\frac{1}{k}$ where k varies from 1 to 4. In sigma notation we could write this as

$$\sum_{k=1}^{k=4} \frac{1}{k}$$

Example

The sum

$$x_1 + x_2 + x_3 + x_4 + \ldots + x_{19} + x_{20}$$

can be written

$$\sum_{k=1}^{k=20} x_k$$

There is nothing special about using the letter k. For example

$$\sum_{n=1}^{n=7} n^2 \quad \text{stands for} \quad 1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 7^2$$

We can also use a little trick to alternate the signs of the numbers between + and −. Note that $(-1)^2 = 1$, $(-1)^3 = -1$ and so on.

Example

Write out fully what is meant by

$$\sum_{i=0}^{5} \frac{(-1)^{i+1}}{2i+1}$$

Solution

$$\sum_{i=0}^{5} \frac{(-1)^{i+1}}{2i+1} = -1 + \frac{1}{3} - \frac{1}{5} + \frac{1}{7} - \frac{1}{9} + \frac{1}{11}$$

Exercises

1. Write out fully what is meant by

   a) $\sum_{i=1}^{i=5} i^2$,
   b) $\sum_{k=1}^{4} (2k+1)^2$,
   c) $\sum_{k=0}^{4} (2k+1)^2$.

2. Write out fully what is meant by

$$\sum_{k=1}^{k=3} (\bar{x} - x_k)$$

3. Sigma notation is often used in statistical calculations. For example the **mean**, $\bar{x}$, of the n quantities x_1, x_2 ... x_n is found by adding them up and dividing the result by n. Show that the mean can be written as

$$\bar{x} = \frac{\sum_{i=1}^{n} x_i}{n}$$

4. Write out fully what is meant by $\sum_{i=1}^{4} \frac{i}{i+1}$.

5. Write out fully what is meant by $\sum_{k=1}^{3} \frac{(-1)^k}{k}$.

Answers

1. a) $1^2 + 2^2 + 3^2 + 4^2 + 5^2$, b) $3^2 + 5^2 + 7^2 + 9^2$, c) $1^2 + 3^2 + 5^2 + 7^2 + 9^2$.

2. $(\bar{x} - x_1) + (\bar{x} - x_2) + (\bar{x} - x_3)$. 4. $\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5}$. 5. $\frac{-1}{1} + \frac{1}{2} + \frac{-1}{3}$ which equals $-1 + \frac{1}{2} - \frac{1}{3}$.

Partial fractions 1

Introduction

An algebraic fraction can often be rewritten as the sum of simpler fractions that are called **partial fractions**. For example, it can be shown that

$$\frac{8x-12}{x^2-2x-3} \quad \text{can be written in partial fractions as} \quad \frac{3}{x-3}+\frac{5}{x+1}$$

This leaflet explains the procedure for finding partial fractions.

1. Proper and improper fractions

When the degree of the numerator, that is the highest power on top, is less than the degree of the denominator, that is the highest power on the bottom, the fraction is said to be **proper**. The fraction

$$\frac{8x-12}{x^2-2x-3}$$

satisfies this condition and so is proper.

If a fraction is not proper it is said to be **improper**. For example, the fraction

$$\frac{2x^3+7x}{x^2+x+1}$$

is improper because the degree of the numerator, 3, is greater than the degree of the denominator, 2.

The first stage in the process of finding partial fractions is to determine whether the fraction is proper or improper because proper fractions are simpler to deal with. Improper fractions are dealt with in leaflet 2.25.

2. Finding partial fractions of proper fractions

You should carry out the following steps:

Step 1

Factorise the denominator if it is not already factorised.

Step 2

When you have factorised the denominator the factors can take various forms. You must study these forms carefully. For example, you may find

$$(3x+2)(x+1)$$

These factors are both referred to as **linear factors**. Generally a linear factor has the form $ax + b$ where a and b are numbers.

The factors could be the same, as in

$$(3x + 2)(3x + 2) \qquad \text{that is} \qquad (3x + 2)^2$$

This is called a **repeated linear factor**. Generally, such a factor has the form $(ax + b)^2$.

Another possible form is

$$x^2 + x + 1$$

This is a **quadratic factor** which cannot be factorised into linear factors. Generally such a factor has the form $ax^2 + bx + c$.

It is essential that you examine the factors carefully to see which type you have. The form that the partial fractions take depends upon the type of factors obtained.

You should examine the factors of the denominator to decide which sorts of partial fraction you will need. These are summarised in the following box.

Each **linear factor**, $ax + b$, produces a partial fraction of the form

$$\frac{A}{ax + b}$$

where A represents an unknown constant which must be found.

A **repeated linear factor**, $(ax + b)^2$, produces two partial fractions of the form

$$\frac{A}{ax + b} + \frac{B}{(ax + b)^2}$$

where A and B represent two unknown constants which must be found.

A **quadratic factor** $ax^2 + bx + c$, which cannot be factorised, produces a partial fraction of the form

$$\frac{Ax + B}{ax^2 + bx + c}$$

Step 3

Find the unknown constants, A, B, ... This is done using a method known as **equating coefficients**, or by substituting specific values for x, or by a mixture of both methods.

These three steps are illustrated in the examples in leaflet 2.24.

Partial fractions 2

1. Worked examples

Example

Express $\dfrac{5x-4}{x^2-x-2}$ as the sum of its partial fractions.

Solution

First we factorise the denominator: $x^2 - x - 2 = (x+1)(x-2)$. Next, examine the form of the factors. The factor $(x+1)$ is a linear factor and produces a partial fraction of the form $\frac{A}{x+1}$. The factor $(x-2)$ is also a linear factor, and produces a partial fraction of the form $\frac{B}{x-2}$. Hence

$$\frac{5x-4}{x^2-x-2} = \frac{5x-4}{(x+1)(x-2)} = \frac{A}{x+1} + \frac{B}{x-2} \qquad (1)$$

where A and B are constants which must be found. Finally we find the constants. Writing the right-hand side using a common denominator we have

$$\frac{5x-4}{(x+1)(x-2)} = \frac{A(x-2)+B(x+1)}{(x+1)(x-2)}$$

The denominators on both sides are the same, and so the numerators on both sides must be the same too. Thus

$$5x - 4 = A(x-2) + B(x+1) \qquad (2)$$

We shall first demonstrate how to find A and B by **substituting specific values for** x. By appropriate choice of the value for x, the right-hand side of Equation 2 can be simplified. For example, letting $x = 2$ we find $6 = A(0) + B(3)$, so that $6 = 3B$, that is $B = 2$. Then by letting $x = -1$ in Equation 2 we find $-9 = A(-3) + B(0)$, from which $-3A = -9$, so that $A = 3$. Substituting these values for A and B into Equation 1 gives

$$\frac{5x-4}{x^2-x-2} = \frac{3}{x+1} + \frac{2}{x-2}$$

The constants can also be found by **equating coefficients**. From Equation 2 we have

$$\begin{aligned} 5x-4 &= A(x-2)+B(x+1) \\ &= Ax - 2A + Bx + B \\ &= (A+B)x + B - 2A \end{aligned}$$

Comparing the coefficients of x on the left- and right-hand sides gives $5 = A + B$. Comparing the constant terms gives $-4 = B - 2A$. These simultaneous equations in A and B can be solved to find $A = 3$ and $B = 2$ as before. Often a combination of the two methods is needed.

Example

Express $\dfrac{2x^2+3}{(x+2)(x+1)^2}$ in partial fractions.

Solution

The denominator is already factorised. Note that there is a linear factor $(x+2)$ and a repeated linear factor $(x+1)^2$. So we can write

$$\frac{2x^2+3}{(x+2)(x+1)^2} = \frac{A}{x+2} + \frac{B}{x+1} + \frac{C}{(x+1)^2} \tag{3}$$

The right-hand side is now written over a common denominator to give

$$\frac{2x^2+3}{(x+2)(x+1)^2} = \frac{A(x+1)^2 + B(x+2)(x+1) + C(x+2)}{(x+2)(x+1)^2}$$

Therefore

$$2x^2 + 3 = A(x+1)^2 + B(x+2)(x+1) + C(x+2) \tag{4}$$

A and C can be found by substituting values for x which simplify the right-hand side. For example if $x = -1$ we find $2(-1)^2 + 3 = A(0) + B(0) + C$ from which $C = 5$. Similarly if we choose $x = -2$ we find $8 + 3 = A(-1)^2 + B(0) + C(0)$ so that $A = 11$. To find B we shall use the method of **equating coefficients**, although we could equally have substituted any other value for x. To equate coefficients we remove the brackets on the right-hand side of Equation 4. After collecting like terms we find that Equation 4 can be written

$$2x^2 + 3 = (A+B)x^2 + (2A + 3B + C)x + (A + 2B + 2C)$$

By comparing the coefficients of x^2 on both sides we see that $(A+B)$ must equal 2. Since we already know $A = 11$, this means $B = -9$. Finally substituting our values of A, B and C into Equation 3 we have $\dfrac{2x^2+3}{(x+2)(x+1)^2} = \dfrac{11}{x+2} - \dfrac{9}{x+1} + \dfrac{5}{(x+1)^2}$.

Exercises

1. Show that $\frac{x-1}{6x^2+5x+1} = \frac{3}{2x+1} - \frac{4}{3x+1}$.

2. Show that $\frac{s+4}{s^2+s} = \frac{4}{s} - \frac{3}{s+1}$.

3. The fraction $\dfrac{5x^2+4x+11}{(x^2+x+4)(x+1)}$ has a quadratic factor in the denominator which cannot be factorised. Thus the required form of the partial fractions is

$$\frac{5x^2+4x+11}{(x^2+x+4)(x+1)} = \frac{Ax+B}{x^2+x+4} + \frac{C}{x+1}$$

Show that $\dfrac{5x^2+4x+11}{(x^2+x+4)(x+1)} = \dfrac{2x-1}{x^2+x+4} + \dfrac{3}{x+1}$.

Partial fractions 3

Introduction

This leaflet describes how the partial fractions of an improper fraction can be found.

1. Partial fractions of improper fractions

An algebraic fraction is improper if the degree (highest power) of the numerator is greater than or equal to that of the denominator. Suppose we let d equal the degree of the denominator, and n the degree of the numerator. Then, in addition to the partial fractions arising from factors in the denominator we must include an additional term: this additional term is a polynomial of degree $n - d$.

Note that:

a polynomial of degree 0 is: A, a constant

a polynomial of degree 1 is: $Ax + B$

a polynomial of degree 2 is: $Ax^2 + Bx + C$,

and so on.

Example

Express $\dfrac{3x^2 + 2x}{x + 1}$ as partial fractions.

Solution

This fraction is improper because $n = 2$ and $d = 1$ and so $n \geq d$. We must include a polynomial of degree $n - d = 1$ as well as the normal partial fractions arising from the factors of the denominator. Thus

$$\frac{3x^2 + 2x}{x + 1} = Ax + B + \frac{C}{x + 1}$$

Writing the right-hand side over a common denominator gives

$$\frac{3x^2 + 2x}{x + 1} = \frac{(Ax + B)(x + 1) + C}{x + 1}$$

and so

$$3x^2 + 2x = (Ax + B)(x + 1) + C$$

As before we can equate coefficients or substitute values for x to find

$$C = 1, A = 3, \text{ and } B = -1$$

Finally

$$\frac{3x^2 + 2x}{x + 1} = 3x - 1 + \frac{1}{x + 1}$$

Example

Express $\dfrac{s^2 + 2s + 1}{s^2 + s + 1}$ in partial fractions.

Solution

Here $n = 2$, and $d = 2$. The fraction is therefore improper, with $n - d = 0$. We must include a polynomial of degree 0, that is a constant, in addition to the usual partial fractions arising from the factors of the denominator. In this example the denominator will not factorise and so this remains a quadratic factor. So,

$$\frac{s^2 + 2s + 1}{s^2 + s + 1} = A + \frac{Bs + C}{s^2 + s + 1}$$

Writing the right-hand side over a common denominator gives

$$\frac{s^2 + 2s + 1}{s^2 + s + 1} = \frac{A(s^2 + s + 1) + (Bs + C)}{s^2 + s + 1}$$

and so

$$s^2 + 2s + 1 = A(s^2 + s + 1) + (Bs + C)$$

Equating coefficients of s^2 shows that $A = 1$. Equating coefficients of s shows that $B = 1$, and you should check that $C = 0$. Hence

$$\frac{s^2 + 2s + 1}{s^2 + s + 1} = 1 + \frac{s}{s^2 + s + 1}$$

Exercises

1. Show that

$$\frac{x^4 + 2x^3 - 2x^2 + 4x - 1}{x^2 + 2x - 3} = x^2 + 1 + \frac{1}{x + 3} + \frac{1}{x - 1}$$

2. Show that

$$\frac{4x^3 + 12x^2 + 13x + 7}{4x^2 + 4x + 1} = x + 2 + \frac{2}{2x + 1} + \frac{3}{(2x + 1)^2}$$

3. Show that

$$\frac{6x^3 + x^2 + 5x - 1}{x^3 + x} = 6 - \frac{1}{x} + \frac{2x - 1}{x^2 + 1}$$

Completing the square

Introduction

In this leaflet we explain a procedure called **completing the square**. This can be used to solve quadratic equations, and is also important in the calculation of some integrals and when it is necessary to find inverse Laplace transforms.

1. Perfect squares

Some quadratic expressions are **perfect squares**. For example

$$x^2 - 6x + 9 \qquad \text{can be written as} \qquad (x-3)^2$$

The equivalence of this pair of expressions is easily verified by squaring $(x-3)$, as in

$$(x-3)(x-3) = x^2 - 3x - 3x + 9 = x^2 - 6x + 9$$

Similarly, $x^2 + 14x + 49$ can be written as $(x+7)^2$. Both $x^2 - 6x + 9$ and $x^2 + 14x + 49$ are perfect squares because they can be written as the square of another expression.

2. Completing the square

In general, a quadratic expression cannot be written in the form $(***)^2$ and so will not be a perfect square. Often, the best we can do is to write a quadratic expression as a perfect square, plus or minus some constant. Doing this is called **completing the square**.

Example
Show that $x^2 + 8x + 7$ can be written as $(x+4)^2 - 9$.

Solution
Squaring the term $(x+4)$ we find

$$\begin{aligned}(x+4)^2 &= (x+4)(x+4)\\ &= x^2 + 8x + 16\end{aligned}$$

So

$$\begin{aligned}(x+4)^2 - 9 &= x^2 + 8x + 16 - 9\\ &= x^2 + 8x + 7\end{aligned}$$

We have shown that $x^2 + 8x + 7$ can be written as a perfect square minus a constant, that is $(x+4)^2 - 9$. We have completed the square.

The following result may help you complete the square, although with practice it is easier to do this by inspection.

$$x^2 + kx + c = (x + \frac{k}{2})^2 - \frac{k^2}{4} + c$$

You can verify this is true by squaring the term in brackets and simplifying the right-hand side.

Example
Complete the square for the expression $x^2 + 6x + 2$.

Solution
Comparing $x^2 + 6x + 2$ with the general form in the box we note that $k = 6$ and $c = 2$. Then

$$\begin{aligned} x^2 + 6x + 2 &= (x + \frac{6}{2})^2 - \frac{6^2}{4} + 2 \\ &= (x + 3)^2 - 7 \end{aligned}$$

and we have completed the square.

Example
Complete the square for the expression $x^2 - 7x + 3$.

Solution
Comparing $x^2 - 7x + 3$ with the general form in the box we note that $k = -7$ and $c = 3$. Then

$$\begin{aligned} x^2 - 7x + 3 &= (x + \frac{-7}{2})^2 - \frac{(-7)^2}{4} + 3 \\ &= (x - \frac{7}{2})^2 - \frac{49}{4} + 3 \\ &= (x - \frac{7}{2})^2 - \frac{37}{4} \end{aligned}$$

and we have completed the square.

Exercises
1. Complete the square for a) $x^2 - 8x + 5$, b) $x^2 + 12x - 7$.

2. Completing the square can be used in the solution of quadratic equations. Complete the square for $x^2 + 8x + 1$ and use your result to solve the equation $x^2 + 8x + 1 = 0$.

3. By first extracting a factor of 3, complete the square for $3x^2 + 6x + 11$.

Answers
1. a) $(x - 4)^2 - 11$, b) $(x + 6)^2 - 43$.

2. $(x + 4)^2 - 15$. Hence the equation can be written $(x + 4)^2 - 15 = 0$ from which $(x + 4)^2 = 15$, $(x + 4) = \pm\sqrt{15}$ and finally $x = -4 \pm \sqrt{15}$.

3. $3x^2 + 6x + 11 = 3[x^2 + 2x + \frac{11}{3}] = 3[(x + 1)^2 + \frac{8}{3}]$.

What is a surd?

Introduction

In engineering calculations, numbers are often given in **surd form**. This leaflet explains what is meant by surd form, and gives some circumstances in which surd forms arise.

1. Surd form

Suppose we wish to simplify $\sqrt{\frac{1}{4}}$. We can write it as $\frac{1}{2}$. On the other hand, some numbers involving roots, such as $\sqrt{2}$, $\sqrt{3}$, $\sqrt[3]{6}$ cannot be expressed exactly in the form of a fraction. Any number of the form $\sqrt[n]{a}$, which cannot be written as a fraction of two integers is called a **surd**.

Whilst numbers like $\sqrt{2}$ have decimal approximations which can be obtained using a calculator, e.g $\sqrt{2} = 1.414\ldots$, we emphasise that these are **approximations**, whereas the form $\sqrt{2}$ is **exact**.

2. Writing surds in equivalent forms

It is often possible to write surds in equivalent forms. To do this you need to be aware that

$$\sqrt{a \times b} = \sqrt{a} \times \sqrt{b}$$

However, be warned that $\sqrt{a+b}$ is not equal to $\sqrt{a} + \sqrt{b}$.

For example $\sqrt{48}$ can be written

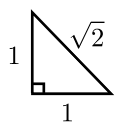
$$\sqrt{3 \times 16} = \sqrt{3} \times \sqrt{16} = 4\sqrt{3}$$

Similarly, $\sqrt{60}$ can be written

$$\sqrt{4 \times 15} = \sqrt{4} \times \sqrt{15} = 2\sqrt{15}$$

3. Applications

Surds arise naturally in a number of applications. For example, by using Pythagoras' theorem we find the length of the hypotenuse of the triangle shown below to be $\sqrt{2}$.



Surds arise in the solution of quadratic equations using the formula. For example, the solution of $x^2 + 8x + 1 = 0$ is obtained as

$$\begin{aligned} x &= \frac{-8 \pm \sqrt{8^2 - 4(1)(1)}}{2} \\ &= \frac{-8 \pm \sqrt{60}}{2} \\ &= \frac{-8 \pm \sqrt{4 \times 15}}{2} \\ &= \frac{-8 \pm 2\sqrt{15}}{2} \\ &= -4 \pm \sqrt{15} \end{aligned}$$

This answer has been left in surd form.

Exercises

1. Write the following in their simplest forms.

a) $\sqrt{63}$, b) $\sqrt{180}$.

2. By multiplying numerator and denominator by $\sqrt{2} + 1$ show that

$$\frac{1}{\sqrt{2} - 1} \qquad \text{is equivalent to} \qquad \sqrt{2} + 1$$

The process of rewriting a fraction in this way, so that all surds appear in the numerator only, is called **rationalisation**.

3. Rationalise the denominator of a) $\frac{1}{\sqrt{2}}$, b) $\frac{1}{\sqrt{5}}$.

4. Simplify $\sqrt{18} - 2\sqrt{2} + \sqrt{8}$.

Answers

1. a) $3\sqrt{7}$, b) $6\sqrt{5}$. 3. a) $\frac{\sqrt{2}}{2}$, b) $\frac{\sqrt{5}}{5}$. 4. $3\sqrt{2}$.

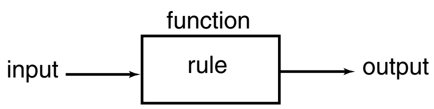
What is a function?

Introduction

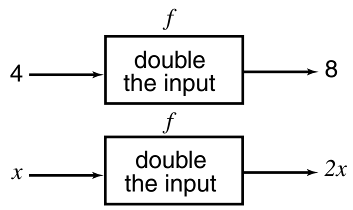
A quantity whose value can change is known as a **variable**. **Functions** are used to describe the rules which define the ways in which such a change can occur. The purpose of this leaflet is to explain functions and their notation.

1. The function rule

A function is a rule which operates on an **input** and produces an **output**. This can be illustrated using a **block diagram** such as that shown below. We can think of the function as a mathematical machine which processes the input, using a given rule, in order to produce an output. We often write the rule inside the box.



In order for a rule to be a function it must produce only a single output for any given input. The function with the rule 'double the input' is shown below.



Note that with an input of 4 the function would produce an output of 8. With a more general input, x say, the output will be $2x$. It is usual to assign a letter or other symbol to a function in order to label it. The doubling function pictured in the example above has been given the symbol f.

A function is a rule which operates on an input and produces a single output from that input.

For the doubling function it is common to use the notation

$$f(x) = 2x$$

This indicates that with an input x, the function, f, produces an output of $2x$. The input to the function is placed in the brackets after the function label 'f'. $f(x)$ is read as 'f is a function of x', or simply 'f of x', meaning that the output from the function depends upon the value of the input x.

Example
State the rule of each of the following functions:

a) $f(x) = 7x + 9$,   b) $h(t) = t^3 + 2$,   c) $p(x) = x^3 + 2$.

Solution
a) The rule for f is 'multiply the input by 7 and then add 9'.

b) The rule for h is 'cube the input and add 2'.

c) The rule for p is 'cube the input and add 2'.

Note from parts b) and c) that it is the rule that is important when describing a function and not the letters being used. Both $h(t)$ and $p(x)$ instruct us to 'cube the input and add 2'.

The input to a function is called its **argument**. We can obtain the output from a function if we are given its argument. For example, given the function $f(x) = 3x + 2$ we may require the value of the output when the argument is 5. We write this as $f(5)$. Here, $f(5) = 3 \times 5 + 2 = 17$.

Example
Given the function $f(x) = 4x + 3$ find a) $f(-1)$,   b) $f(6)$.

Solution
a) Here the argument is -1. We find $f(-1) = 4 \times (-1) + 3 = -1$.

b) $f(6) = 4(6) + 3 = 27$.

Sometimes the argument will be an algebraic expression, as in the following example.

Example
Given the function $y(x) = 5x - 3$ find

a) $y(t)$,   b) $y(7t)$,   c) $y(z + 2)$.

Solution
The function rule is multiply the input by 5, and subtract 3. We can apply this rule whatever the argument.

a) To find $y(t)$ multiply the argument, t, by 5 and subtract 3 to give $y(t) = 5t - 3$.

b) Now the argument is $7t$. So $y(7t) = 5(7t) - 3 = 35t - 3$.

c) In this case the argument is $z + 2$. We find $y(z + 2) = 5(z + 2) - 3 = 5z + 10 - 3 = 5z + 7$.

Exercises
1. Write down a function which can be used to describe the following rules:

a) 'cube the input and divide the result by 2',   b) 'divide the input by 5 and then add 7'.

2. Given the function $f(x) = 7x - 3$ find a) $f(3)$,   b) $f(6)$,   c) $f(-2)$.

3. If $g(t) = t^2$ write down expressions for a) $g(x)$,   b) $g(3t)$,   c) $g(x + 4)$.

Answers
1. a) $f(x) = \frac{x^3}{2}$,   b) $f(x) = \frac{x}{5} + 7$.   2. a) 18,   b) 39,   c) -17.

3. a) $g(x) = x^2$,   b) $g(3t) = (3t)^2 = 9t^2$,   c) $g(x + 4) = (x + 4)^2 = x^2 + 8x + 16$.

Engineering Maths First Aid Kit

3.2

The graph of a function

Introduction

A very useful pictorial representation of a function is the **graph**. In this leaflet we remind you of important conventions when graph plotting.

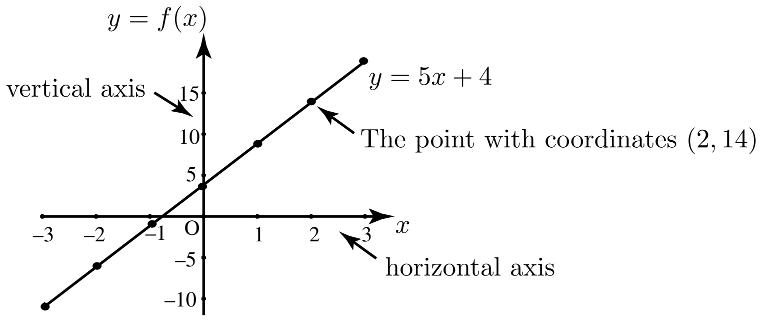
1. The graph of a function

Consider the function $f(x) = 5x + 4$.

We can choose several values for the input and calculate the corresponding outputs. We have done this for integer values of x between -3 and 3 and the results are shown in the table.

x	-3	-2	-1	0	1	2	3
$f(x)$	-11	-6	-1	4	9	14	19

To plot the graph we first draw a pair of **axes** – a vertical axis and a horizontal axis. These are drawn at right-angles to each other and intersect at the **origin** O as shown below.



Each pair of input and output values can be represented on a graph by a single point. The input values are measured along the horizontal axis and the output values along the vertical axis. A uniform scale is drawn on each axis sufficient to accommodate all the required points. The points plotted in this way are then joined together, in this case by a straight line. This is the graph of the function. Each point on the graph can be represented by a pair of **coordinates** in the form $(x, f(x))$. Each axis should be labelled to show its variable.

2. Dependent and independent variables

The horizontal axis is often called the x axis. The vertical axis is commonly referred to as the y axis. So, we often write the function above, not as $f(x) = 5x + 4$, but rather as

$$y = 5x + 4$$

Since x and y can have a number of different values they are variables. Here x is called the **independent variable** and y is called the **dependent variable**. Knowing or choosing a value of the independent variable, x, the function allows us to calculate the corresponding value of the dependent variable, y. To show this dependence we often write $y(x)$. This notation simply means that y depends upon x. Note that it is the independent variable which is the input to the function and the dependent variable which is the output.

Example
Consider the function given by $y = 2t^2 + 1$, for values of t between -2 and 2.

a) State the independent variable.

b) State the dependent variable.

c) Plot a graph of the function.

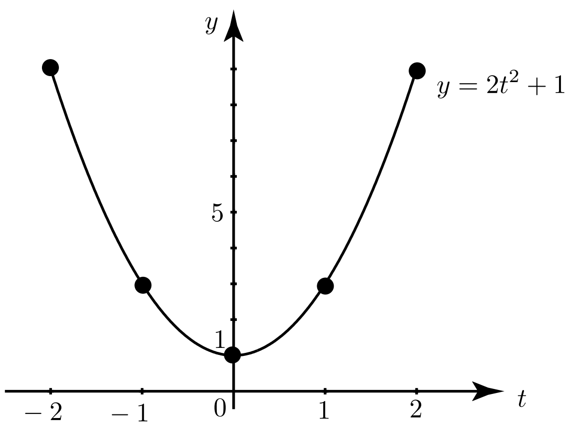
Solution
a) The independent variable is t.

b) The dependent variable is y.

c) A table of input and output values should be constructed first. Such a table is shown below.

t	-2	-1	0	1	2
y	9	3	1	3	9

Each pair of t and y values in the table is plotted as a single point. The points are then joined with a smooth curve to produce the required graph as shown below.



Exercises
1. Plot a graph of each of the following functions. In each case state the dependent and independent variables.

a) $y = f(x) = 3x + 2$, for x between -2 and 5, b) $y = f(t) = 6 - t^2$, for t between 1 and 5.

Answers
1. a) dependent variable is y, independent variable is x.

b) dependent variable is y, independent variable is t.

Engineering Maths First Aid Kit

3.3

The straight line

Introduction

Straight line graphs arise in many engineering applications. This leaflet discusses the mathematical equation which describes a straight line and explains the terms 'gradient' and 'intercept'.

1. The equation of a straight line

Any equation of the form

$$y = mx + c$$

where m and c are fixed numbers, (i.e. constants), has a graph which is a straight line.

For example,

$$y = 3x + 5, \qquad y = \frac{2}{3}x + 8 \qquad \text{and} \qquad y = -3x - 7$$

all have graphs which are straight lines, but

$$y = 3x^2 + 4, \qquad y = \frac{2}{3x} - 7, \qquad \text{and} \qquad y = -14\sqrt{x}$$

have graphs which are not straight lines. The essential feature of a straight line equation is that x and y occur only to the power 1.

2. The straight line graph

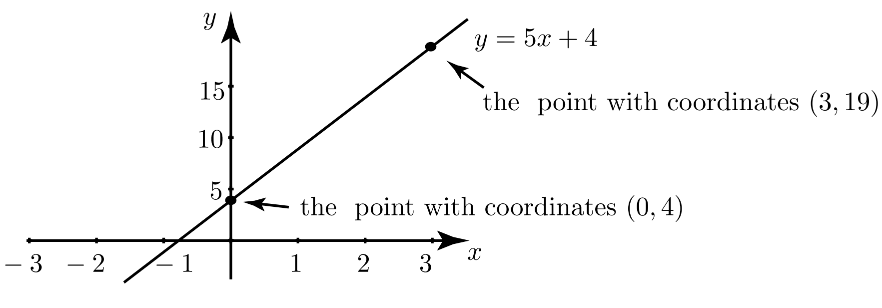
Any straight line graph can be plotted very simply by finding just two points which lie on the line and joining them. It is a good idea to find a third point just as a check.

Example

Plot a graph of the straight line with equation $y = 5x + 4$.

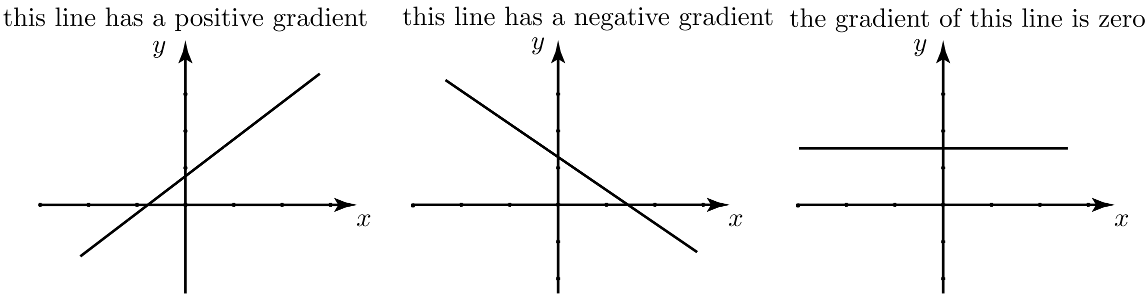
Solution

From the equation, note that when $x = 0$, the value of y is 4. Similarly when $x = 3$, $y = 19$. So the points $(0, 4)$ and $(3, 19)$ lie on the graph. These points are plotted and joined together to form the straight line graph.

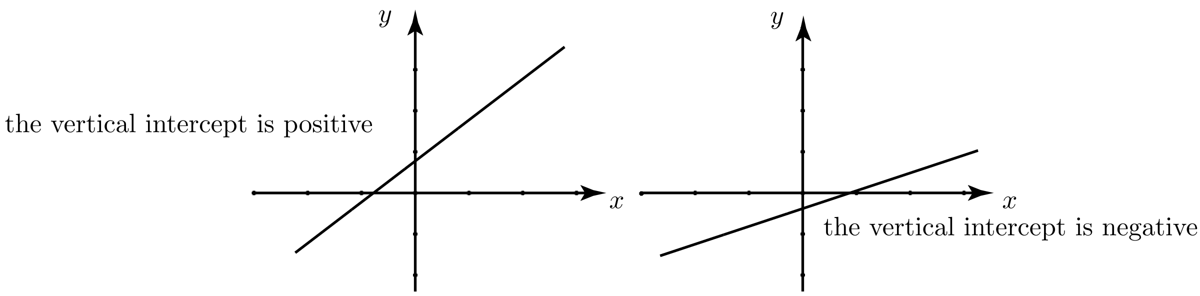


3. The gradient and intercept of a straight line

In the equation $y = mx + c$ the value of m is called the **gradient** of the line. It can be positive, negative or zero. Lines with a positive gradient slope upwards, from left to right. Lines with a negative gradient slope downwards from left to right. Lines with a zero gradient are horizontal.



The value of c is called the **vertical intercept** of the line. It is the value of y when $x = 0$. When drawing a line, c gives the position where the line cuts the vertical axis.



Example
Determine the gradient and vertical intercept of each line.

a) $y = 12x - 6$, b) $y = 5 - 2x$, c) $4x - y + 13 = 0$, d) $y = 8$, e) $y = 4x$.

Solution
a) Comparing $y = 12x - 6$ with $y = mx + c$ we see that $m = 12$, so the gradient of the line is 12. The fact that this is positive means that the line slopes upwards as we move from left to right. The vertical intercept is -6. This line cuts the vertical axis below the horizontal axis.

b) Comparing $y = 5 - 2x$ with $y = mx + c$ we see that $m = -2$, so the gradient is -2. The line slopes downwards as we move from left to right. The vertical intercept is 5.

c) We write $4x - y + 13 = 0$ in standard form as $y = 4x + 13$ and note that $m = 4$, $c = 13$.

d) Comparing $y = 8$ with $y = mx + c$ we see that $m = 0$ and $c = 8$. This line is horizontal.

e) Comparing $y = 4x$ with $y = mx + c$ we see that $m = 4$ and $c = 0$.

Exercises
1. State the gradient and intercept of each of the following lines.

a) $y = 5x + 6$, b) $y = 3x - 11$, c) $y = -2x + 7$, d) $y = 9$, e) $y = 7 - x$.

Answers
1. a) gradient 5, intercept 6 b) 3,-11, c) -2,7, d) 0,9, e) -1, 7.

Engineering Maths First Aid Kit

3.4

The exponential constant e

Introduction

The letter e is used in many mathematical calculations to stand for a particular number known as the **exponential constant**. This leaflet provides information about this important constant, and the related **exponential function**.

1. The exponential constant

The exponential constant is an important mathematical constant and is given the symbol e. Its value is approximately 2.718. It has been found that this value occurs so frequently when mathematics is used to model physical and chemical phenomena that it is convenient to write simply e.

It is often necessary to work out powers of this constant, such as e^2, e^3 and so on. Your scientific calculator will be programmed to do this already. You should check that you can use your calculator to do this. Look for a button marked e^x, and check that

$$e^2 = 7.389, \qquad \text{and} \qquad e^3 = 20.086$$

In both cases we have quoted the answer to three decimal places although your calculator will give a more accurate answer than this.

You should also check that you can evaluate negative and fractional powers of e such as

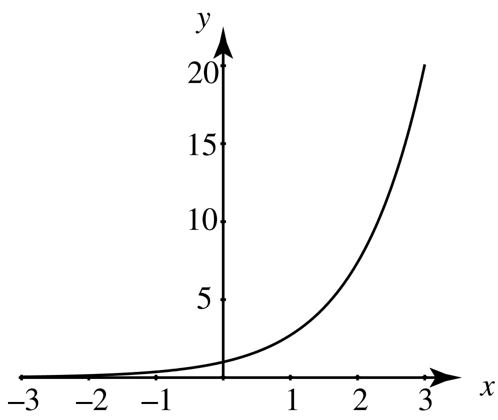
$$e^{1/2} = 1.649 \qquad \text{and} \qquad e^{-2} = 0.135$$

2. The exponential function

If we write $y = e^x$ we can calculate the value of y as we vary x. Values obtained in this way can be placed in a table. For example:

x	-3	-2	-1	0	1	2	3
$y = e^x$	0.050	0.135	0.368	1	2.718	7.389	20.086

This is a table of values of the **exponential function** e^x. If pairs of x and y values are plotted we obtain a **graph** of the exponential function as shown overleaf. If you have never seen this function before it will be a worthwhile exercise to plot it for yourself.



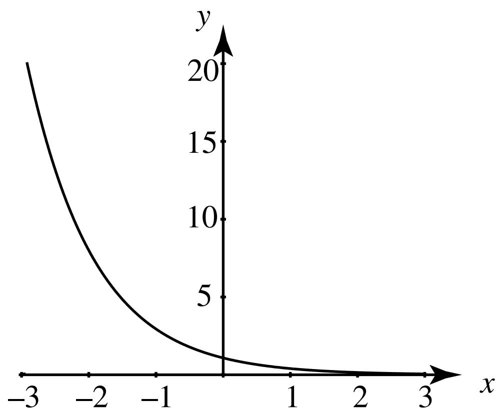
A graph of the exponential function $y = \mathrm{e}^x$

It is important to note that as x becomes larger, the value of e^x grows without bound. We write this mathematically as $\mathrm{e}^x \to \infty$ as $x \to \infty$. This behaviour is known as **exponential growth**.

3. The negative exponential function

A related function is the **negative exponential function** $y = \mathrm{e}^{-x}$. A table of values of this function is shown below together with its graph.

x	-3	-2	-1	0	1	2	3
$y = \mathrm{e}^{-x}$	20.086	7.389	2.718	1	0.368	0.135	0.050



A graph of the negative exponential function $y = \mathrm{e}^{-x}$

It is very important to note that as x becomes larger, the value of e^{-x} approaches zero. We write this mathematically as $\mathrm{e}^{-x} \to 0$ as $x \to \infty$. This behaviour is known as **exponential decay**.

Exercises

A useful exercise would be to draw up tables of values and plot graphs of some related functions:

a) $y = \mathrm{e}^{2x}$, b) $y = \mathrm{e}^{0.5x}$, c) $y = -\mathrm{e}^{x}$, d) $y = -\mathrm{e}^{-x}$, e) $y = 1 - \mathrm{e}^{-x}$.

The hyperbolic functions

Introduction

In a number of applications, the exponential functions e^x and e^{-x} occur in particular combinations and these combinations are referred to as the **hyperbolic functions**. This leaflet defines these functions and shows their graphs.

1. The hyperbolic functions

The **hyperbolic cosine** is defined as

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

The **hyperbolic sine** is defined as

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

These are often referred to as the 'cosh' function and the 'shine' function. They are nothing more than combinations of the exponential functions e^x and e^{-x}.

Your scientific calculator can be used to evaluate these functions. Usually the 'hyp cos' and 'hyp sin' buttons are used. You may need to refer to your calculator manual. Check that you can use your calculator by verifying that

$$\sinh 3 = 10.018 \qquad \text{and} \qquad \cosh 4.2 = 33.351$$

You may like to verify that the same values can be obtained by using the exponential functions, that is

$$\sinh 3 = \frac{e^3 - e^{-3}}{2} \qquad \text{and} \qquad \cosh 4.2 = \frac{e^{4.2} + e^{-4.2}}{2}$$

The **hyperbolic tangent** is defined as

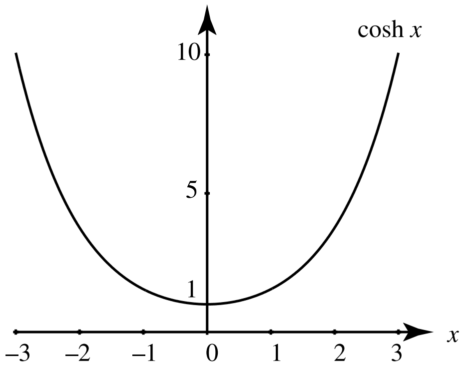
$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

Other hyperbolic functions are

$$\text{sech}\, x = \frac{1}{\cosh x}, \qquad \text{cosech}\, x = \frac{1}{\sinh x}, \qquad \coth x = \frac{\cosh x}{\sinh x} = \frac{1}{\tanh x}$$

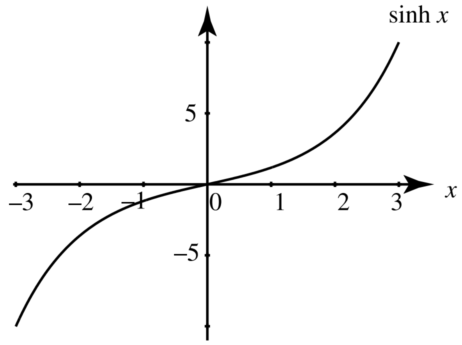
By drawing up tables of values, or indeed by using the properties of the exponential functions, graphs can be plotted. The graphs of $\cosh x$, $\sinh x$ and $\tanh x$ are shown below.

2. Graphs of the hyperbolic functions



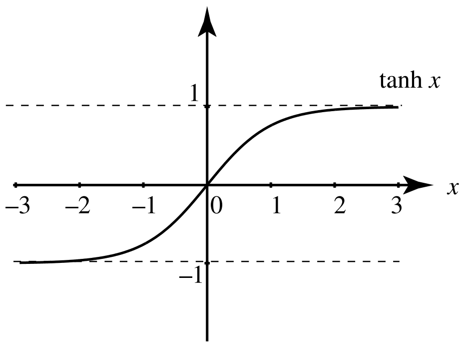
Some properties of $\cosh x$

- $\cosh 0 = 1$ and $\cosh x$ is greater than 1 for all other values of x
- the graph is symmetrical about the y axis. Mathematically this means $\cosh(-x) = \cosh x$. Cosh x is said to be an **even function**.
- $\cosh x \to +\infty$ as $x \to \pm\infty$



Some properties of $\sinh x$

- $\sinh 0 = 0$, the graph passes through the origin
- $\sinh(-x) = -\sinh x$. Sinh x is said to be an **odd function** – it has rotational symmetry about the origin.
- $\sinh x \to +\infty$ as $x \to +\infty$, $\quad \sinh x \to -\infty$ as $x \to -\infty$



Some properties of $\tanh x$

- $\tanh 0 = 0$ and $-1 < \tanh x < 1$ for all x
- $\tanh(-x) = -\tanh x$. Tanh x is said to be an **odd function** – it has rotational symmetry about the origin.
- $\tanh x \to +1$ as $x \to +\infty$, $\qquad \tanh x \to -1$ as $x \to -\infty$

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3.6

The hyperbolic identities

Introduction

The hyperbolic functions satisfy a number of identities. These allow expressions involving the hyperbolic functions to be written in different, yet equivalent forms. Several commonly used identities are given in this leaflet.

1. Hyperbolic identities

$$\cosh x = \frac{\mathrm{e}^x + \mathrm{e}^{-x}}{2}, \qquad \sinh x = \frac{\mathrm{e}^x - \mathrm{e}^{-x}}{2}$$
$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{\mathrm{e}^x - \mathrm{e}^{-x}}{\mathrm{e}^x + \mathrm{e}^{-x}}$$
$$\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{\mathrm{e}^x + \mathrm{e}^{-x}}$$
$$\operatorname{cosech} x = \frac{1}{\sinh x} = \frac{2}{\mathrm{e}^x - \mathrm{e}^{-x}}$$
$$\coth x = \frac{\cosh x}{\sinh x} = \frac{1}{\tanh x} = \frac{\mathrm{e}^x + \mathrm{e}^{-x}}{\mathrm{e}^x - \mathrm{e}^{-x}}$$

$$\cosh^2 x - \sinh^2 x = 1$$
$$1 - \tanh^2 x = \operatorname{sech}^2 x$$
$$\coth^2 x - 1 = \operatorname{cosech}^2 x$$

$$\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$$
$$\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$$
$$\tanh(x \pm y) = \frac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y}$$
$$\sinh 2x = 2 \sinh x \cosh x$$
$$\cosh 2x = \cosh^2 x + \sinh^2 x$$
$$\cosh^2 x = \frac{\cosh 2x + 1}{2}$$
$$\sinh^2 x = \frac{\cosh 2x - 1}{2}$$

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3.7

The logarithm function

Introduction

This leaflet provides a table of values and graph of the logarithm function $y = \log_e x$.

1. The logarithm function and its graph

Logarithms have been explained in leaflet *2.19 What is a logarithm?*. There we showed how logarithms provide alternative ways of writing expressions involving powers, and we showed how a calculator can be used to find logarithms.

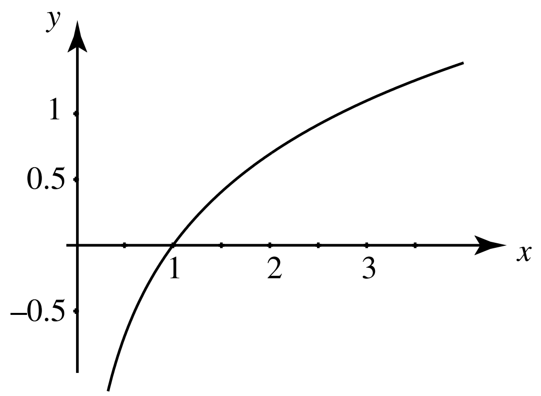
The **natural logarithm function** is $y = \log_e x$, also written $\ln x$.
Note that we have chosen to use logarithms to base e as this is the most common base.

Using a calculator it is possible to construct a table of values of $y = \log_e x$ as follows:

x	0.5	1	1.5	2	2.5	3	3.5
$y = \log_e x$	-0.693	0	0.405	0.693	0.916	1.099	1.253

You should check these values for yourself to make sure that you can obtain them.

If pairs of x and y values are plotted we obtain a **graph** of the logarithm function as shown.



The graph of the natural logarithm $y = \log_e x$

Note that the logarithm function is only defined for positive values of x. We cannot find the logarithm of 0, or the logarithm of a negative number.

As an exercise you should draw up a similar table for the function $y = \log_{10} x$ and plot its graph. The graph should have the same general shape as the one above although most of the points on the graph are different.

Solving equations involving logarithms and exponentials

Introduction

It is often necessary to solve an equation in which the unknown occurs as a power, or exponent. For example, you may need to find the value of x which satisfies $2^x = 32$. Very often the base will be the exponential constant e, as in the equation $e^x = 20$. To understand what follows you must be familiar with the exponential constant. See leaflet *3.4 The exponential constant* if necessary.

You will also come across equations involving logarithms. For example you may need to find the value of x which satisfies $\log_{10} x = 34$. You will need to understand what is meant by a logarithm, and the laws of logarithms (leaflets *2.19 What is a logarithm?* and *2.20 The laws of logarithms*). In this leaflet we explain how such equations can be solved.

1. Revision of logarithms

Logarithms provide an alternative way of writing expressions involving powers. If

$$a = b^c \qquad \text{then} \qquad \log_b a = c$$

For example: $100 = 10^2$ can be written as $\log_{10} 100 = 2$.

Similarly, $e^3 = 20.086$ can be written as $\log_e 20.086 = 3$.

The third law of logarithms states that, for logarithms of any base,

$$\log A^n = n \log A$$

For example, we can write $\log_{10} 5^2$ as $2 \log_{10} 5$, and $\log_e 7^3$ as $3 \log_e 7$.

2. Solving equations involving powers

Example
Solve the equation $e^x = 14$.

Solution
Writing $e^x = 14$ in its alternative form using logarithms we obtain $x = \log_e 14$, which can be evaluated directly using a calculator to give 2.639.

Example
Solve the equation $e^{3x} = 14$.

Solution
Writing $e^{3x} = 14$ in its alternative form using logarithms we obtain $3x = \log_e 14 = 2.639$. Hence $x = \frac{2.639}{3} = 0.880$.

To solve an equation of the form $2^x = 32$ it is necessary to take the logarithm of both sides of the equation. This is referred to as 'taking logs'. Usually we use logarithms to base 10 or base e because values of these logarithms can be obtained using a scientific calculator.

Starting with $2^x = 32$, then taking logs produces $\log_{10} 2^x = \log_{10} 32$. Using the third law of logarithms, we can rewrite the left-hand side to give $x \log_{10} 2 = \log_{10} 32$. Dividing both sides by $\log_{10} 2$ gives

$$x = \frac{\log_{10} 32}{\log_{10} 2}$$

The right-hand side can now be evaluated using a calculator in order to find x:

$$x = \frac{\log_{10} 32}{\log_{10} 2} = \frac{1.5051}{0.3010} = 5$$

Hence $2^5 = 32$. Note that this answer can be checked by substitution into the original equation.

3. Solving equations involving logarithms

Example
Solve the equation $\log_{10} x = 0.98$.

Solution
Rewriting the equation in its alternative form using powers gives $10^{0.98} = x$. A calculator can be used to evaluate $10^{0.98}$ to give $x = 9.550$.

Example
Solve the equation $\log_e 5x = 1.7$.

Solution
Rewriting the equation in its alternative form using powers gives $e^{1.7} = 5x$. A calculator can be used to evaluate $e^{1.7}$ to give $5x = 5.4739$ so that $x = 1.095$ to 3dp.

Exercises
1. Solve each of the following equations to find x.

a) $3^x = 15$, b) $e^x = 15$, c) $3^{2x} = 9$, d) $e^{5x-1} = 17$, e) $10^{3x} = 4$.

2. Solve the equations a) $\log_e 2x = 1.36$, b) $\log_{10} 5x = 2$, c) $\log_{10}(5x + 3) = 1.2$.

Answers
1. a) 2.465, b) 2.708, c) 1, d) 0.767, e) 0.201.
2. a) 1.948 (3dp), b) 20, c) 2.570 (3dp).

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3.9

Polar coordinates

Introduction

An alternative to using (x, y), or cartesian coordinates, is to use 'polar coordinates'. These are particularly useful for problems involving circular symmetry. This leaflet explains polar coordinates and shows how it is possible to convert between cartesian and polar coordinates.

1. Polar coordinates

When you were first introduced to coordinate systems you will have used *cartesian coordinates*. These are the standard x and y coordinates of a point, P, such as that shown in Figure 3.9.1a where the x axis is horizontal, the y axis is vertical and their intersection is the *origin*, O.

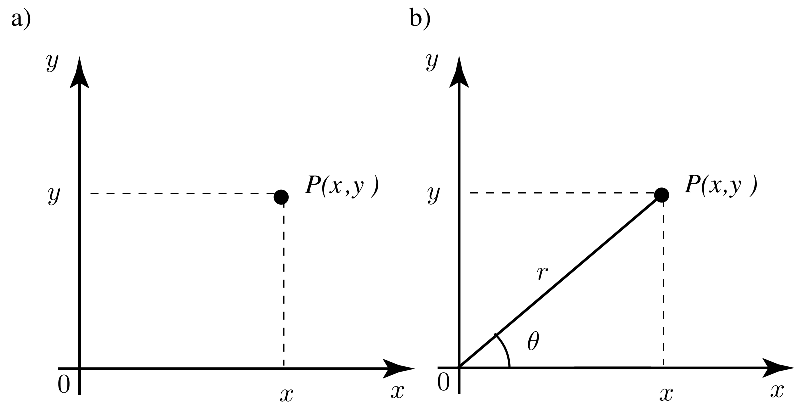


Figure 3.9.1. a) Cartesian coordinates, b) Polar coordinates

The position of any point in the plane can be described uniquely by giving its x and y coordinates.

An alternative way of describing the position of a point is to draw a line from the origin to the point as shown in Figure 3.9.1b. We can then state the length of this line, r, and the angle, θ, between the positive direction of the x axis and the line. These quantities are called the **polar coordinates** of P. It is conventional to denote the polar coordinates of a point either in the form (r, θ) or $r\angle\theta$, although the latter is preferred to avoid confusion with cartesian coordinates. When measuring the angle θ we use the convention that positive angles are measured anticlockwise, and negative angles are measured clockwise. The length of OP is always taken to be positive. Figure 3.9.2 shows several points and their polar coordinates.

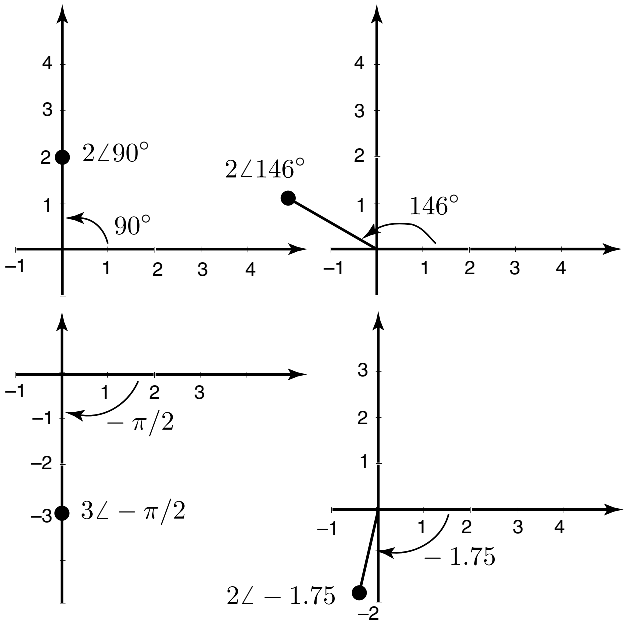


Figure 3.9.2. Some points and their polar coordinates

2. Conversion between cartesian and polar coordinates

Look back at Figure 3.9.1b. From trigonometry note that $\cos\theta = \dfrac{x}{r}$ so that $x = r\cos\theta$. Similarly $\sin\theta = \dfrac{y}{r}$ so that $y = r\sin\theta$. Hence if we know the polar coordinates of a point $r\angle\theta$, we can find its cartesian coordinates.

Alternatively, using Pythagoras' theorem note that $r = \sqrt{x^2+y^2}$. Further, $\tan\theta = \dfrac{y}{x}$ so that $\theta = \tan^{-1}\left(\dfrac{y}{x}\right)$. However, when calculating θ you should take special care to ensure that θ is located in the correct quadrant. The result produced by your calculator can be misleading. A diagram should always be sketched and will help you decide the correct quadrant.

$$x = r\cos\theta, \qquad y = r\sin\theta$$
$$r = \sqrt{x^2+y^2}, \qquad \tan\theta = \frac{y}{x}$$

Exercises

In each case sketch a diagram showing the point in question. Angles in degrees are denoted by the degrees symbol °. Otherwise assume that the angle is measured in radians.

1. Calculate the cartesian coordinates of the following points.

a) $3\angle 2$, b) $4\angle 0.7$, c) $1\angle 180°$.

2. Calculate the polar coordinates of the following points.

a) $(3, 4)$, b) $(-2, 1)$, c) $(-2, -3)$.

Answers

1. a) $(-1.25, 2.73)$, b) $(3.06, 2.58)$, c) $(-1, 0)$. 2. a) $5\angle 0.927$, b) $\sqrt{5}\angle 2.678$, c) $\sqrt{13}\angle - 2.159$

Degrees and radians

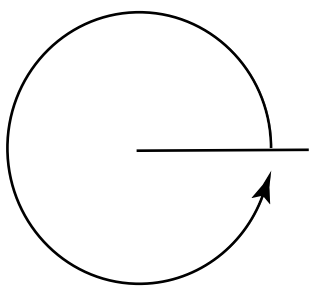
Introduction

Angles can be measured in units of either degrees or radians. This leaflet explains these units and shows how it is possible to convert between them.

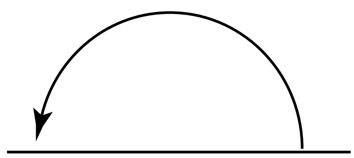
1. Degrees and radians

Angles can be measured in units of either degrees or radians. The symbol for degree is °. Usually no symbol is used to denote radians.

A complete revolution is defined as 360° or 2π radians. π stands for the number 3.14159 . . . and you can work with this if you prefer. However, in many calculations you will find that you need to work directly with multiples of π.



complete revolution = $360° = 2\pi$ radians



half a revolution = $180° = \pi$ radians

It is easy to use the fact that $360° = 2\pi$ radians to convert between the two measures. We have

$$\begin{aligned} 360° &= 2\pi \text{ radians} \\ 1° &= \frac{2\pi}{360} = \frac{\pi}{180} \text{ radians} \\ 1 \text{ radian} &= \frac{180}{\pi} \text{ degrees} \approx 57.3° \end{aligned}$$

Example

a) Convert 65° to radians. b) Convert 1.75 radians to degrees.

Solution

a)

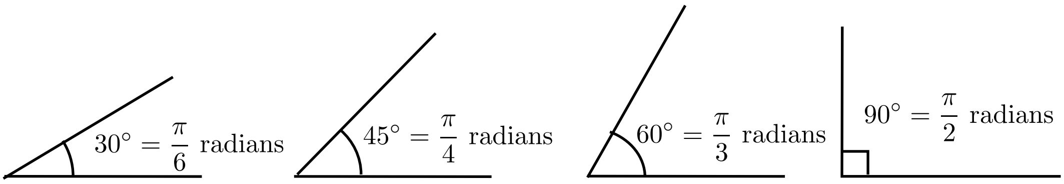
$$
\begin{aligned}
1^\circ &= \frac{\pi}{180} \text{ radians} \\
65^\circ &= 65 \times \frac{\pi}{180} \\
&= 1.134 \text{ radians}
\end{aligned}
$$

b)

$$
\begin{aligned}
1 \text{ radian} &= \frac{180}{\pi} \text{ degrees} \\
1.75 \text{ radians} &= 1.75 \times \frac{180}{\pi} \\
&= 100.268^\circ
\end{aligned}
$$

Note the following commonly met angles:

$$
\begin{array}{lll}
30^\circ = \frac{\pi}{6} \text{ radians} & 45^\circ = \frac{\pi}{4} \text{ radians} & 60^\circ = \frac{\pi}{3} \text{ radians} \\
90^\circ = \frac{\pi}{2} \text{ radians} & 135^\circ = \frac{3\pi}{4} \text{ radians} & 180^\circ = \pi \text{ radians}
\end{array}
$$



Your calculator should be able to work with angles measured in both radians and degrees. Usually the MODE button allows you to select the appropriate measure. When calculations involve calculus you should always work with radians and not degrees.

Exercises

1. Convert each of the following angles given in degrees, to radians. Give your answers correct to 2 decimal places.

a) 32°, b) 95°, c) 217°.

2. Convert each of the following angles given in radians, to degrees. Give your answers correct to 2 decimal places.

a) 3 radians, b) 2.4 radians, c) 1 radian.

3. Convert each of the following angles given in radians, to degrees. Do not use a calculator.

a) $\frac{\pi}{15}$, b) $\frac{\pi}{5}$.

4. Convert the following angles given in degrees, to radians. Do not use a calculator and give your answers as multiples of π.

a) 90°, b) 72°, c) -45°.

Answers

1. a) 0.56 radians, b) 1.66 radians, c) 3.79 radians. 2. a) 171.89°, b) 137.51°, c) 57.30°.
3. a) 12°, b) 36°. 4. a) $\frac{\pi}{2}$ radians, b) $\frac{2\pi}{5}$ radians, c) $-\frac{\pi}{4}$ radians.

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4.2

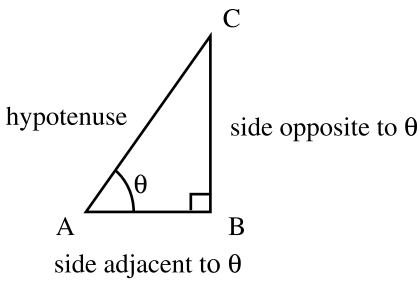
The trigonometrical ratios

Introduction

The trigonometrical ratios sine, cosine and tangent appear frequently in many engineering problems. This leaflet revises the meaning of these terms.

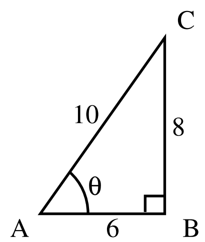
1. Sine, cosine and tangent ratios

Study the right-angled triangle ABC shown below.



The side opposite the right-angle is called the **hypotenuse**. The side **opposite** to θ is BC. The remaining side, AB, is said to be **adjacent** to θ.

Suppose we know the lengths of each of the sides as in the figure below.



We can then divide the length of one side by the length of one of the other sides.

The ratio $\frac{BC}{AC}$ is known as the **sine** of angle θ. This is abbreviated to $\sin\theta$. In the triangle shown we see that

$$\sin\theta = \frac{8}{10} = 0.8$$

The ratio $\frac{AB}{AC}$ is known as the **cosine** of angle θ. This is abbreviated to $\cos\theta$. In the triangle shown we see that

$$\cos\theta = \frac{6}{10} = 0.6$$

The ratio $\frac{BC}{AB}$ is known as the **tangent** of angle θ. This is abbreviated to $\tan\theta$. In the triangle shown we see that

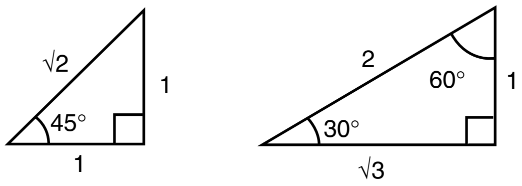
$$\tan\theta = \frac{8}{6} = 1.3333$$

In any right-angled triangle we define the trigonometrical ratios as follows:

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{BC}{AC} \qquad \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{AB}{AC}$$

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{BC}{AB}$$

2. Some standard, or common, triangles



$$\sin 45° = \frac{1}{\sqrt{2}}, \qquad \cos 45° = \frac{1}{\sqrt{2}}, \qquad \tan 45° = 1$$

$$\sin 30° = \frac{1}{2}, \qquad \cos 30° = \frac{\sqrt{3}}{2}, \qquad \tan 30° = \frac{1}{\sqrt{3}}$$

$$\sin 60° = \frac{\sqrt{3}}{2}, \qquad \cos 60° = \frac{1}{2}, \qquad \tan 60° = \sqrt{3}$$

3. Using a calculator

If we know the angles in a right-angled triangle the trigonometrical ratios can be found using a scientific calculator. Look for the sine, cosine and tangent buttons on your calculator and make sure that you can use them by verifying that

$$\sin 50° = 0.7660, \qquad \cos 32° = 0.8480$$

Your calculator will be able to handle angles measured in either radians or degrees. It will be necessary for you to choose the appropriate units. Study your calculator manual to learn how to do this. Check that

$$\sin 0.56 \text{ radians} = 0.5312, \qquad \tan 1.4 \text{ radians} = 5.7979$$

4. Finding an angle when a trigonometrical ratio is known

If we are given, or know, a value for $\sin\theta$, $\cos\theta$ or $\tan\theta$ we may want to work out the corresponding angle θ. This process is known as finding the inverse sine, inverse cosine or inverse tangent. Your calculator will be pre-programmed for doing this. The buttons will be labelled invsin, or $\sin^{-1}$, and so on.

Check that you can use your calculator to show that if $\sin\theta = 0.75$ then $\theta = 48.59°$.

Mathematically we write this as follows:

$$\text{if } \sin\theta = 0.75, \quad \text{then } \theta = \sin^{-1} 0.75 = 48.59°$$

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4.3

Graphs of the trigonometric functions

Introduction

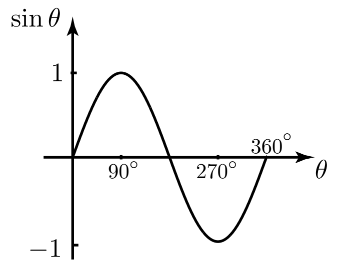
The trigonometric functions play a very important role in engineering mathematics. Familiarity with the graphs of these functions is essential. Graphs of the trigonometric functions sine, cosine and tangent, together with some tabulated values are shown here for reference.

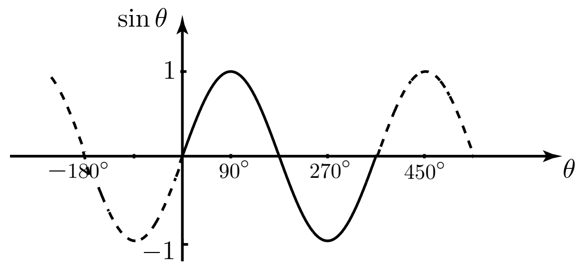
1. The sine function

Using a scientific calculator a table of values of $\sin\theta$ can be drawn up as θ varies from 0 to 360°.

θ	0°	30°	60°	90°	120°	150°
$\sin\theta$	0	0.5000	0.8660	1	0.8660	0.5000
θ	180°	210°	240°	270°	300°	360°
$\sin\theta$	0	−0.5000	−0.8660	−1	−0.8660	0

Using the table, a graph of the function $y = \sin\theta$ can be plotted and is shown below on the left.





If further values, outside the range 0° to 360°, are calculated we find that the wavy pattern repeats itself as shown on the right. We say that the sine function is **periodic** with period 360°. Some values are particularly important and should be remembered:

$$\sin 0° = 0, \qquad \sin 90° = 1, \qquad \sin 180° = 0, \qquad \sin 270° = -1$$

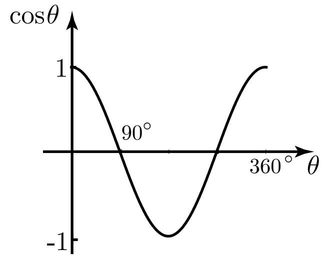
The maximum value of $\sin\theta$ is 1, and the minimum value is −1.

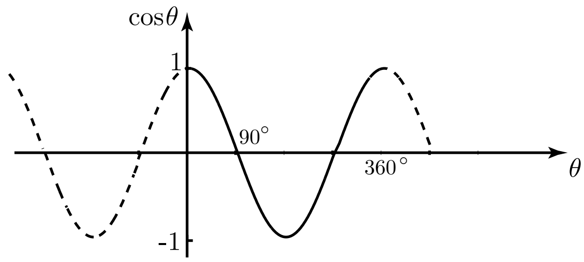
2. The cosine function

θ	$0°$	$30°$	$60°$	$90°$	$120°$	$150°$
$\cos\theta$	1	0.8660	0.5000	0	-0.5000	-0.8660
θ	$180°$	$210°$	$240°$	$270°$	$300°$	$360°$
$\cos\theta$	-1	-0.8660	-0.5000	0	0.5000	1

Using a scientific calculator a table of values of $\cos\theta$ can be drawn up as θ varies from 0 to 360°.

Using the table, a graph of the function $y = \cos\theta$ can be plotted as shown on the left.





If further values are calculated outside the range $0 \le \theta \le 360°$ we find that the wavy pattern repeats itself as shown on the right. We say that the cosine function is periodic with period 360°. Some values are particularly important and should be remembered:

$$\cos 0° = 1, \qquad \cos 90° = 0, \qquad \cos 180° = -1, \qquad \cos 270° = 0$$

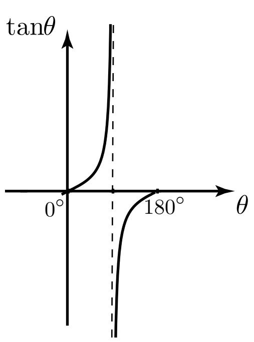
The maximum value of $\cos\theta$ is 1, and the minimum value is -1.

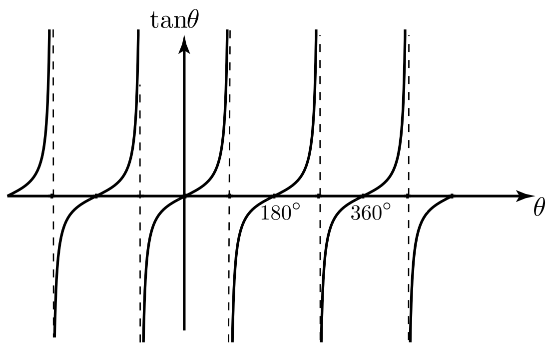
3. The tangent function

Using a scientific calculator a table of values of $\tan\theta$ can be drawn up as θ varies from 0 to 180° although when $\theta = 90°$ you will find that this function is not defined.

θ	0	$45°$	$90°$	$135°$	$180°$
$\tan\theta$	0	1	∞	-1	0

Using the table, a graph of the function $y = \tan\theta$ can be plotted and is shown below on the left.





If further values are calculated outside the range $0 \le \theta \le 180°$ we find that the pattern repeats itself as shown on the right. We say that the tangent function is periodic with period 180°.

Some values are particularly important and should be remembered:

$$\tan 0° = 0, \qquad \tan 45° = 1$$

There is no maximum value of $\tan\theta$ because it increases without bound. There is no minimum value. However there are certain values where $\tan\theta$ is not defined, including $-90°$, $90°$, $270°$ and so on. Here the graph shoots off to infinity.

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4.4

Trigonometrical identities

Introduction

Very often it is necessary to rewrite expressions involving sines, cosines and tangents in alternative forms. To do this we use formulas known as **trigonometric identities**. A number of commonly used identities are listed here.

1. The identities

$$\tan A = \frac{\sin A}{\cos A} \qquad \sec A = \frac{1}{\cos A} \qquad \text{cosec}\, A = \frac{1}{\sin A} \qquad \cot A = \frac{\cos A}{\sin A} = \frac{1}{\tan A}$$

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

$$2 \sin A \cos B = \sin(A + B) + \sin(A - B)$$

$$2 \cos A \cos B = \cos(A - B) + \cos(A + B)$$

$$2 \sin A \sin B = \cos(A - B) - \cos(A + B)$$

$$\sin^2 A + \cos^2 A = 1$$

$$1 + \cot^2 A = \text{cosec}^2 A, \quad \tan^2 A + 1 = \sec^2 A$$

$$\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$$

$$\sin 2A = 2 \sin A \cos A$$

$$\sin^2 A = \frac{1-\cos 2A}{2}, \qquad \cos^2 A = \frac{1+\cos 2A}{2}$$

$$\sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

$$\sin A - \sin B = 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$

$$\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

$$\cos A - \cos B = 2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{B-A}{2}\right)$$

Note: $\sin^2 A$ is the notation used for $(\sin A)^2$. Similarly $\cos^2 A$ means $(\cos A)^2$ and so on.

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4.5

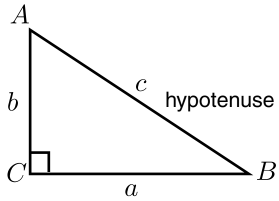
Pythagoras' theorem

Introduction

Pythagoras' theorem relates the lengths of the sides of a right-angled triangle. This leaflet reminds you of the theorem and provides some revision examples and exercises.

1. Pythagoras' theorem

Study the right-angled triangle shown.

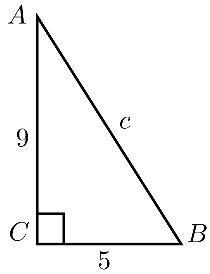


In any right-angled triangle, ABC, the side opposite the right-angle is called the **hypotenuse**. Here we use the convention that the side opposite angle A is labelled a. The side opposite B is labelled b and the side opposite C is labelled c.

Pythagoras' theorem states that the square of the hypotenuse, (c^2), is equal to the sum of the squares of the other two sides, ($a^2 + b^2$).

Pythagoras' theorem: $c^2 = a^2 + b^2$

Example



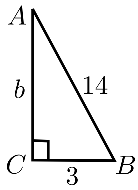
Suppose $AC = 9$ cm and $BC = 5$ cm as shown. Find the length of the hypotenuse, AB.

Solution
Here, $a = BC = 5$, and $b = AC = 9$. Using the theorem

$$\begin{aligned} c^2 &= a^2 + b^2 \\ &= 5^2 + 9^2 \\ &= 25 + 81 \\ &= 106 \\ c &= \sqrt{106} = 10.30 \qquad \text{(2dp.)} \end{aligned}$$

The hypotenuse has length 10.30 cm.

Example
In triangle ABC shown, suppose that the length of the hypotenuse is 14 cm and that $a = BC = 3$ cm. Find the length of AC.



Solution
Here $a = BC = 3$, and $c = AB = 14$. Using the theorem

$$\begin{aligned} c^2 &= a^2 + b^2 \\ 14^2 &= 3^2 + b^2 \\ 196 &= 9 + b^2 \\ b^2 &= 196 - 9 \\ &= 187 \\ b &= \sqrt{187} = 13.67 \qquad \text{(2dp)} \end{aligned}$$

The length of AC is 13.67 cm.

Exercises
1. In triangle ABC in which $C = 90^\circ$, $AB = 25$ cm and $AC = 17$ cm. Find the length BC.
2. In triangle ABC, the angle at B is the right-angle. If $AB = BC = 5$ cm find AC.
3. In triangle CDE the right-angle is E. If $CD = 55$ cm and $DE = 37$ cm find EC.

Answers
1. 18.33 cm. (2dp)
2. $AC = \sqrt{50} = 7.07$ cm. (2dp)
3. $EC = \sqrt{1656} = 40.69$ cm. (2dp)

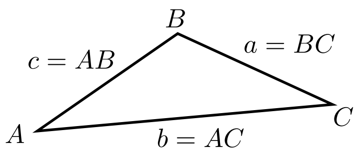
The sine rule and cosine rule

Introduction

To **solve** a triangle is to find the lengths of each of its sides and all its angles. The **sine rule** is used when we are given either a) two angles and one side, or b) two sides and a non-included angle. The **cosine rule** is used when we are given either a) three sides or b) two sides and the included angle.

1. The sine rule

Study the triangle ABC shown below. Let B stands for the angle at B. Let C stand for the angle at C and so on. Also, let $b = AC$, $a = BC$ and $c = AB$.



The sine rule:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Example
In triangle ABC, $B = 21°$, $C = 46°$ and $AB = 9$ cm. Solve this triangle.

Solution
We are given two angles and one side and so the sine rule can be used. Furthermore, since the angles in any triangle must add up to 180° then angle A must be 113°. We know that $c = AB = 9$. Using the sine rule

$$\frac{a}{\sin 113°} = \frac{b}{\sin 21°} = \frac{9}{\sin 46°}$$

So,

$$\frac{b}{\sin 21°} = \frac{9}{\sin 46°}$$

from which

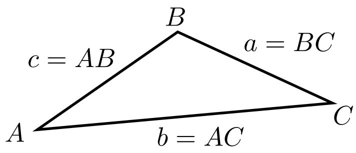
$$b = \sin 21° \times \frac{9}{\sin 46°} = 4.484 \text{ cm} \qquad \text{(3dp)}$$

Similarly

$$a = \sin 113° \times \frac{9}{\sin 46°} = 11.517 \text{ cm} \qquad \text{(3dp)}$$

2. The cosine rule

Refer to the triangle shown below.



The cosine rule:

$$a^2 = b^2 + c^2 - 2bc\cos A, \qquad b^2 = a^2 + c^2 - 2ac\cos B, \qquad c^2 = a^2 + b^2 - 2ab\cos C$$

Example
In triangle ABC, $AB = 42$ cm, $BC = 37$ cm and $AC = 26$ cm. Solve this triangle.

Solution
We are given three sides of the triangle and so the cosine rule can be used. Writing $a = 37$, $b = 26$ and $c = 42$ we have

$$a^2 = b^2 + c^2 - 2bc\cos A$$

from which

$$37^2 = 26^2 + 42^2 - 2(26)(42)\cos A$$

$$\cos A = \frac{26^2 + 42^2 - 37^2}{(2)(26)(42)} = \frac{1071}{2184} = 0.4904$$

and so

$$A = \cos^{-1} 0.4904 = 60.63°$$

You should apply the same technique to verify that $B = 37.76°$ and $C = 81.61°$. You should also check that the angles you obtain add up to 180°.

Exercises
1. Solve the triangle ABC in which $AC = 105$ cm, $AB = 76$ cm and $A = 29°$.

2. Solve the triangle ABC given $C = 40°$, $b = 23$ cm and $c = 19$ cm.

Answers
1. $a = 53.31$cm, $B = 72.72°$, $C = 78.28°$.

2. $A = 11.09°$, $B = 128.91°$, $a = 5.69$ cm.
   or $A = 88.91°$, $B = 51.09°$, $BC = 29.55$ cm.

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5.1

Determinants

Introduction

Determinants are mathematical objects which have applications in engineering mathematics. For example, they can be used in the solution of simultaneous equations, and to evaluate vector products. This leaflet will show you how to calculate the value of a determinant.

1. Evaluating a determinant

The symbol $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ represents the expression $ad - bc$ and is called a **determinant**.

For example

$$\begin{vmatrix} 3 & 2 \\ 1 & 4 \end{vmatrix} \qquad \text{means} \qquad 3 \times 4 \quad - \quad 2 \times 1 = 12 - 2 = 10$$

Because $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ has two rows and two columns we describe it as a '2 by 2' or second-order determinant. Its value is given by

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

If we are given values for a, b, c and d we can use this to calculate the value of the determinant. Note that, once we have worked it out, a determinant is a single number.

Exercises

Evaluate the following determinants:

a) $\begin{vmatrix} 3 & 4 \\ 6 & 5 \end{vmatrix}$ b) $\begin{vmatrix} 2 & -2 \\ 1 & 4 \end{vmatrix}$, c) $\begin{vmatrix} 8 & 5 \\ -2 & 4 \end{vmatrix}$, d) $\begin{vmatrix} 6 & 10 \\ -3 & -5 \end{vmatrix}$, e) $\begin{vmatrix} x & 5 \\ y & 2 \end{vmatrix}$.

Answers

a) $15 - 24 = -9$, b) $8 - (-2) = 10$, c) $32 - (-10) = 42$, d) $-30 - (-30) = 0$, e) $2x - 5y$.

2. Third-order determinants

A third-order or '3 by 3' determinant can be written

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

One way in which it can be evaluated is to use second-order determinants as follows:

$$a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

Note in particular the way that the signs alternate between + and −.

For example

$$\begin{aligned}\begin{vmatrix} 1 & 2 & 1 \\ -1 & 3 & 4 \\ 5 & 1 & 2 \end{vmatrix} &= 1\begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix} - 2\begin{vmatrix} -1 & 4 \\ 5 & 2 \end{vmatrix} + 1\begin{vmatrix} -1 & 3 \\ 5 & 1 \end{vmatrix} \\ &= 1(2) - 2(-22) + 1(-16) \\ &= 2 + 44 - 16 \\ &= 30\end{aligned}$$

Exercises

1. Evaluate each of the following determinants.

a) $\begin{vmatrix} 2 & 4 & 1 \\ 1 & 0 & 4 \\ 5 & -1 & 3 \end{vmatrix}$, b) $\begin{vmatrix} 0 & -3 & 2 \\ -9 & 4 & 1 \\ 6 & 0 & 2 \end{vmatrix}$, c) $\begin{vmatrix} 7 & -2 & 3 \\ -1 & -4 & -4 \\ 6 & -2 & 12 \end{vmatrix}$, d) $\begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix}$.

2. Evaluate each of the following determinants.

a) $\begin{vmatrix} 9 & 12 & 1 \\ 1 & 4 & 1 \\ 1 & 5 & 3 \end{vmatrix}$, b) $\begin{vmatrix} 3 & 12 & 1 \\ -3 & 4 & 1 \\ 4 & 5 & 3 \end{vmatrix}$, c) $\begin{vmatrix} 3 & 9 & 1 \\ -3 & 1 & 1 \\ 4 & 1 & 3 \end{vmatrix}$, d) $\begin{vmatrix} 3 & 9 & 12 \\ -3 & 1 & 4 \\ 4 & 1 & 5 \end{vmatrix}$.

Answers

1. a) 75, b) −120, c) −290, d) abc. 2. a) 40, b) 146, c) 116, d) 198.

3. Fourth-order determinants

These are evaluated using third-order determinants. Once again note the alternating plus and minus sign.

Example

$$\begin{aligned}\begin{vmatrix} 5 & 2 & 6 & 3 \\ 3 & 9 & 12 & 1 \\ -3 & 1 & 4 & 1 \\ 4 & 1 & 5 & 3 \end{vmatrix} &= 5\begin{vmatrix} 9 & 12 & 1 \\ 1 & 4 & 1 \\ 1 & 5 & 3 \end{vmatrix} - 2\begin{vmatrix} 3 & 12 & 1 \\ -3 & 4 & 1 \\ 4 & 5 & 3 \end{vmatrix} + 6\begin{vmatrix} 3 & 9 & 1 \\ -3 & 1 & 1 \\ 4 & 1 & 3 \end{vmatrix} - 3\begin{vmatrix} 3 & 9 & 12 \\ -3 & 1 & 4 \\ 4 & 1 & 5 \end{vmatrix} \\ &= 5(40) - 2(146) + 6(116) - 3(198) \\ &= 200 - 292 + 696 - 594 \\ &= 10\end{aligned}$$

Determinants can be used in the solution of simultaneous equations using Cramer's rule – see the leaflet *5.2 Cramer's rule*.

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5.2

Cramer's rule

Introduction

Cramer's rule is a method for solving linear simultaneous equations. It makes use of determinants and so a knowledge of these is necessary before proceeding.

1. Cramer's rule – two equations

If we are given a pair of simultaneous equations

$$\begin{aligned} a_1x + b_1y &= d_1 \\ a_2x + b_2y &= d_2 \end{aligned}$$

then x and y can be found from

$$x = \frac{\begin{vmatrix} d_1 & b_1 \\ d_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \qquad y = \frac{\begin{vmatrix} a_1 & d_1 \\ a_2 & d_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

Example
Solve the equations

$$\begin{aligned} 3x + 4y &= -14 \\ -2x - 3y &= 11 \end{aligned}$$

Solution
Using Cramer's rule we can write the solution as the ratio of two determinants.

$$x = \frac{\begin{vmatrix} -14 & 4 \\ 11 & -3 \end{vmatrix}}{\begin{vmatrix} 3 & 4 \\ -2 & -3 \end{vmatrix}} = \frac{-2}{-1} = 2, \qquad y = \frac{\begin{vmatrix} 3 & -14 \\ -2 & 11 \end{vmatrix}}{\begin{vmatrix} 3 & 4 \\ -2 & -3 \end{vmatrix}} = \frac{5}{-1} = -5$$

The solution of the simultaneous equations is then $x = 2$, $y = -5$.

2. Cramer's rule – three equations

For the case of three equations in three unknowns: If

$$\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned}$$

then x, y and z can be found from

$$x = \frac{\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}} \qquad y = \frac{\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}} \qquad z = \frac{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}$$

Exercises

Use Cramer's rule to solve the following sets of simultaneous equations.

a)

$$\begin{aligned} 7x + 3y &= 15 \\ -2x + 5y &= -16 \end{aligned}$$

b)

$$\begin{aligned} x + 2y + 3z &= 17 \\ 3x + 2y + z &= 11 \\ x - 5y + z &= -5 \end{aligned}$$

Answers

a) $x = 3$, $y = -2$, b) $x = 1$, $y = 2$, $z = 4$.

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5.3

Multiplying matrices

Introduction

One of the most important operations carried out with matrices is **matrix multiplication**. At first sight this is done in a rather strange way. The reason for this only becomes apparent when matrices are used to solve equations.

1. Some simple examples

To multiply $\begin{pmatrix} 3 & 7 \end{pmatrix}$ by $\begin{pmatrix} 2 \\ 9 \end{pmatrix}$ perform the following calculation.

$$\begin{pmatrix} 3 & 7 \end{pmatrix} \begin{pmatrix} 2 \\ 9 \end{pmatrix} = 3 \times 2 \quad + \quad 7 \times 9 = 6 + 63 = 69$$

Note that we have paired elements in the row of the first matrix with elements in the column of the second matrix, multiplied the paired elements together and added the results.

Another, larger example:

$$\begin{pmatrix} 4 & 2 & 5 \end{pmatrix} \begin{pmatrix} 3 \\ 6 \\ 8 \end{pmatrix} = 4 \times 3 \quad + \quad 2 \times 6 \quad + \quad 5 \times 8 = 12 + 12 + 40 = 64$$

Exercises

1. Evaluate the following:

a) $\begin{pmatrix} 4 & 5 \end{pmatrix} \begin{pmatrix} 2 \\ 9 \end{pmatrix}$, b) $\begin{pmatrix} -3 & 7 \end{pmatrix} \begin{pmatrix} 2 \\ 9 \end{pmatrix}$, c) $\begin{pmatrix} 1 & 3 & 5 \end{pmatrix} \begin{pmatrix} 3 \\ 7 \\ 8 \end{pmatrix}$, d) $\begin{pmatrix} -4 & 2 & 5 \end{pmatrix} \begin{pmatrix} 3 \\ 6 \\ -8 \end{pmatrix}$.

Answers

1. a) 53, b) 57, c) 64, d) −40.

2. More general matrix multiplication

When we multiplied matrices in the previous section the answers were always single numbers. Usually however, the result of multiplying two matrices is another matrix. Two matrices can only be multiplied together if the number of columns in the first matrix is the same as the number of rows in the second. So, if the first matrix has size $p \times q$, that is, it has p rows and q columns, and the second has size $r \times s$, that is, it has r rows and s columns, we can only multiply them together if $q = r$. When this is so, the result of multiplying them together is a $p \times s$ matrix.

Two matrices can only ever be multiplied together if
the number of columns in the first is the same as the number of rows in the second.

Example

Find $\begin{pmatrix} 3 & 7 \\ 4 & 5 \end{pmatrix}\begin{pmatrix} 2 \\ 9 \end{pmatrix}$.

Solution

The first matrix has size 2×2. The second has size 2×1. Clearly the number of columns in the first is the same as the number of rows in the second. So, multiplication is possible and the result will be a 2×1 matrix. The calculation is performed using the same operations as in the examples in the previous section.

$$\begin{pmatrix} 3 & 7 \\ 4 & 5 \end{pmatrix}\begin{pmatrix} 2 \\ 9 \end{pmatrix} = \begin{pmatrix} * \\ * \end{pmatrix}$$

To obtain the first entry in the solution, ignore the second row of the first matrix. You have already seen the required calculations.

$$\begin{pmatrix} 3 & 7 \\ & \end{pmatrix}\begin{pmatrix} 2 \\ 9 \end{pmatrix} = \begin{pmatrix} 69 \\ \end{pmatrix}$$

To obtain the second entry in the solution, ignore the first row of the first matrix.

$$\begin{pmatrix} & \\ 4 & 5 \end{pmatrix}\begin{pmatrix} 2 \\ 9 \end{pmatrix} = \begin{pmatrix} \\ 53 \end{pmatrix}$$

Putting it all together

$$\begin{pmatrix} 3 & 7 \\ 4 & 5 \end{pmatrix}\begin{pmatrix} 2 \\ 9 \end{pmatrix} = \begin{pmatrix} 69 \\ 53 \end{pmatrix}$$

Example

Find $\begin{pmatrix} 2 & 4 \\ 5 & 3 \end{pmatrix}\begin{pmatrix} 3 & 6 \\ -1 & 9 \end{pmatrix}$.

Solution

The first matrix has size 2×2. The second matrix has size 2×2. Clearly the number of columns in the first is the same as the number of rows in the second. The multiplication can be performed and the result will be a 2×2 matrix.

$$\begin{pmatrix} 2 & 4 \\ 5 & 3 \end{pmatrix}\begin{pmatrix} 3 & 6 \\ -1 & 9 \end{pmatrix} = \begin{pmatrix} 2\times 3+4\times(-1) & 2\times 6+4\times 9 \\ 5\times 3+3\times(-1) & 5\times 6+3\times 9 \end{pmatrix} = \begin{pmatrix} 2 & 48 \\ 12 & 57 \end{pmatrix}$$

Exercises

1. Evaluate the following.

a) $\begin{pmatrix} -3 & 2 \\ 3 & 11 \end{pmatrix}\begin{pmatrix} 3 \\ -1 \end{pmatrix}$, b) $\begin{pmatrix} 4 & 2 \\ 5 & 11 \end{pmatrix}\begin{pmatrix} 3 & 10 \\ -1 & 9 \end{pmatrix}$, c) $\begin{pmatrix} 2 & 1 \\ 1 & 9 \end{pmatrix}\begin{pmatrix} 2 & 0 & 2 \\ 5 & 13 & 1 \end{pmatrix}$.

Answers

1. a) $\begin{pmatrix} -11 \\ -2 \end{pmatrix}$, b) $\begin{pmatrix} 10 & 58 \\ 4 & 149 \end{pmatrix}$, c) $\begin{pmatrix} 9 & 13 & 5 \\ 47 & 117 & 11 \end{pmatrix}$.

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5.4

The inverse of a 2×2 matrix

Introduction

Once you know how to multiply matrices it is natural to ask whether they can be divided. The answer is no. However, by defining another matrix called the **inverse matrix** it is possible to work with an operation which plays a similar role to division. In this leaflet we explain what is meant by an inverse matrix and how the inverse of a 2×2 matrix is calculated.

1. The inverse of a 2×2 matrix

The **inverse** of a 2×2 matrix, A, is another 2×2 matrix denoted by A^{-1} with the property that

$$AA^{-1} = A^{-1}A = I$$

where I is the 2×2 identity matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. That is, multiplying a matrix by its inverse produces an identity matrix. Note that in this context A^{-1} does not mean $\frac{1}{A}$.

Not all 2×2 matrices have an inverse matrix. If the determinant of the matrix is zero, then it will not have an inverse, and the matrix is said to be **singular**. Only non-singular matrices have inverses.

2. A simple formula for the inverse

In the case of a 2×2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ a simple formula exists to find its inverse:

$$\text{if} \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{then} \quad A^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Example

Find the inverse of the matrix $A = \begin{pmatrix} 3 & 1 \\ 4 & 2 \end{pmatrix}$.

Solution

Using the formula

$$\begin{aligned} A^{-1} &= \frac{1}{(3)(2) - (1)(4)}\begin{pmatrix} 2 & -1 \\ -4 & 3 \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} 2 & -1 \\ -4 & 3 \end{pmatrix} \end{aligned}$$

This could be written as

$$\begin{pmatrix} 1 & -\frac{1}{2} \\ -2 & \frac{3}{2} \end{pmatrix}$$

You should check that this answer is correct by performing the matrix multiplication AA^{-1}. The result should be the identity matrix $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Example

Find the inverse of the matrix $A = \begin{pmatrix} 2 & 4 \\ -3 & 1 \end{pmatrix}$.

Solution

Using the formula

$$\begin{aligned} A^{-1} &= \frac{1}{(2)(1) - (4)(-3)} \begin{pmatrix} 1 & -4 \\ 3 & 2 \end{pmatrix} \\ &= \frac{1}{14} \begin{pmatrix} 1 & -4 \\ 3 & 2 \end{pmatrix} \end{aligned}$$

This can be written

$$A^{-1} = \begin{pmatrix} 1/14 & -4/14 \\ 3/14 & 2/14 \end{pmatrix} = \begin{pmatrix} 1/14 & -2/7 \\ 3/14 & 1/7 \end{pmatrix}$$

although it is quite permissible to leave the factor $\frac{1}{14}$ at the front of the matrix.

Exercises

1. Find the inverse of $A = \begin{pmatrix} 1 & 5 \\ 3 & 2 \end{pmatrix}$.

2. Explain why the inverse of the matrix $\begin{pmatrix} 6 & 4 \\ 3 & 2 \end{pmatrix}$ cannot be calculated.

3. Show that $\begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix}$ is the inverse of $\begin{pmatrix} 3 & -4 \\ -2 & 3 \end{pmatrix}$.

Answers

1. $A^{-1} = \frac{1}{-13} \begin{pmatrix} 2 & -5 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} -\frac{2}{13} & \frac{5}{13} \\ \frac{3}{13} & -\frac{1}{13} \end{pmatrix}$.

2. The determinant of the matrix is zero, that is, it is singular and so has no inverse.

The inverse of a matrix

Introduction

In this leaflet we explain what is meant by an inverse matrix and how it is calculated.

1. The inverse of a matrix

The **inverse** of a square $n \times n$ matrix, A, is another $n \times n$ matrix denoted by A^{-1} such that

$$AA^{-1} = A^{-1}A = I$$

where I is the $n \times n$ identity matrix. That is, multiplying a matrix by its inverse produces an identity matrix. Not all square matrices have an inverse matrix. If the determinant of the matrix is zero, then it will not have an inverse, and the matrix is said to be **singular**. Only non-singular matrices have inverses.

2. A formula for finding the inverse

Given any non-singular matrix A, its inverse can be found from the formula

$$A^{-1} = \frac{\text{adj}\, A}{|A|}$$

where $\text{adj}\, A$ is the **adjoint matrix** and $|A|$ is the determinant of A. The procedure for finding the adjoint matrix is given below.

3. Finding the adjoint matrix

The adjoint of a matrix A is found in stages:

(1) Find the transpose of A, which is denoted by A^T. The transpose is found by interchanging the rows and columns of A. So, for example, the first column of A is the first row of the transposed matrix; the second column of A is the second row of the transposed matrix, and so on.

(2) The **minor** of any element is found by covering up the elements in its row and column and finding the determinant of the remaining matrix. By replacing each element of A^T by its minor, we can write down a *matrix of minors* of A^T.

(3) The **cofactor** of any element is found by taking its minor and imposing a **place sign** according to the following rule

$$\begin{pmatrix} + & - & + & \dots \\ - & + & - & \dots \\ + & - & + & \dots \\ \dots & \dots & \dots & \ddots \end{pmatrix}$$

This means, for example, that to find the cofactor of an element in the first row, second column, the sign of the minor is changed. On the other hand to find the cofactor of an element in the second row, second column, the sign of the minor is unaltered. This is equivalent to multiplying the minor by '+1' or '−1' depending upon its position. In this way we can form a *matrix of cofactors* of A^T. This matrix is called the **adjoint** of A, denoted adj A.

The matrix of cofactors of the transpose of A is called the adjoint matrix, adj A

This procedure may seem rather cumbersome, so it is illustrated now by means of an example.

Example

Find the adjoint, and hence the inverse, of $A = \begin{pmatrix} 1 & -2 & 0 \\ 3 & 1 & 5 \\ -1 & 2 & 3 \end{pmatrix}$.

Solution

Follow the stages outlined above. First find the transpose of A by taking the first column of A to be the first row of A^T, and so on:

$$A^T = \begin{pmatrix} 1 & 3 & -1 \\ -2 & 1 & 2 \\ 0 & 5 & 3 \end{pmatrix}$$

Now find the minor of each element in A^T. The minor of the element '1' in the first row, first column, is obtained by covering up the elements in its row and column to give $\begin{pmatrix} 1 & 2 \\ 5 & 3 \end{pmatrix}$ and finding the determinant of this, which is −7. The minor of the element '3' in the second column of the first row is found by covering up elements in its row and column to give $\begin{pmatrix} -2 & 2 \\ 0 & 3 \end{pmatrix}$ which has determinant −6. We continue in this fashion and form a new matrix by replacing every element of A^T by its minor. Check for yourself that this process gives

$$\text{matrix of minors of } A^T = \begin{pmatrix} -7 & -6 & -10 \\ 14 & 3 & 5 \\ 7 & 0 & 7 \end{pmatrix}$$

Then impose the place sign. This results in the matrix of cofactors, that is, the adjoint of A.

$$\text{adj } A = \begin{pmatrix} -7 & 6 & -10 \\ -14 & 3 & -5 \\ 7 & 0 & 7 \end{pmatrix}$$

Notice that to complete this last stage, each element in the matrix of minors has been multiplied by 1 or −1 according to its position.

It is a straightforward matter to show that the determinant of A is 21. Finally

$$A^{-1} = \frac{\text{adj } A}{|A|} = \frac{1}{21}\begin{pmatrix} -7 & 6 & -10 \\ -14 & 3 & -5 \\ 7 & 0 & 7 \end{pmatrix}$$

Exercise

1. Show that the inverse of $\begin{pmatrix} 1 & 3 & 2 \\ 0 & 5 & 1 \\ -1 & 3 & 0 \end{pmatrix}$ is $\frac{1}{4}\begin{pmatrix} -3 & 6 & -7 \\ -1 & 2 & -1 \\ 5 & -6 & 5 \end{pmatrix}$.

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5.6

Using the inverse matrix to solve equations

Introduction

One of the most important applications of matrices is to the solution of linear simultaneous equations. In this leaflet we explain how this can be done.

1. Writing simultaneous equations in matrix form

Consider the simultaneous equations

$$\begin{aligned} x + 2y &= 4 \\ 3x - 5y &= 1 \end{aligned}$$

Provided you understand how matrices are multiplied together you will realise that these can be written in matrix form as

$$\begin{pmatrix} 1 & 2 \\ 3 & -5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$$

Writing

$$A = \begin{pmatrix} 1 & 2 \\ 3 & -5 \end{pmatrix}, \qquad X = \begin{pmatrix} x \\ y \end{pmatrix}, \qquad \text{and} \qquad B = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$$

we have

$$AX = B$$

This is the **matrix form** of the simultaneous equations. Here the unknown is the matrix X, since A and B are already known. A is called the **matrix of coefficients**.

2. Solving the simultaneous equations

Given

$$AX = B$$

we can multiply both sides by the inverse of A, provided this exists, to give

$$A^{-1}AX = A^{-1}B$$

But $A^{-1}A = I$, the identity matrix. Furthermore, $IX = X$, because multiplying any matrix by an identity matrix of the appropriate size leaves the matrix unaltered. So

$$X = A^{-1}B$$

$$\text{if} \quad AX = B, \qquad \text{then} \qquad X = A^{-1}B$$

This result gives us a method for solving simultaneous equations. All we need do is write them in matrix form, calculate the inverse of the matrix of coefficients, and finally perform a matrix multiplication.

Example
Solve the simultaneous equations

$$\begin{aligned} x + 2y &= 4 \\ 3x - 5y &= 1 \end{aligned}$$

Solution

We have already seen these equations in matrix form:

$$\begin{pmatrix} 1 & 2 \\ 3 & -5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$$

We need to calculate the inverse of $A = \begin{pmatrix} 1 & 2 \\ 3 & -5 \end{pmatrix}$.

$$\begin{aligned} A^{-1} &= \frac{1}{(1)(-5) - (2)(3)} \begin{pmatrix} -5 & -2 \\ -3 & 1 \end{pmatrix} \\ &= -\frac{1}{11} \begin{pmatrix} -5 & -2 \\ -3 & 1 \end{pmatrix} \end{aligned}$$

Then X is given by

$$\begin{aligned} X = A^{-1}B &= -\frac{1}{11} \begin{pmatrix} -5 & -2 \\ -3 & 1 \end{pmatrix} \begin{pmatrix} 4 \\ 1 \end{pmatrix} \\ &= -\frac{1}{11} \begin{pmatrix} -22 \\ -11 \end{pmatrix} \\ &= \begin{pmatrix} 2 \\ 1 \end{pmatrix} \end{aligned}$$

Hence $x = 2$, $y = 1$ is the solution of the simultaneous equations.

Exercises
1. Solve the following sets of simultaneous equations using the inverse matrix method.

a) $\begin{aligned} 5x + y &= 13 \\ 3x + 2y &= 5 \end{aligned}$, b) $\begin{aligned} 3x + 2y &= -2 \\ x + 4y &= 6 \end{aligned}$.

Answers

1. a) $x = 3, y = -2$, b) $x = -2, y = 2$.

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6.1

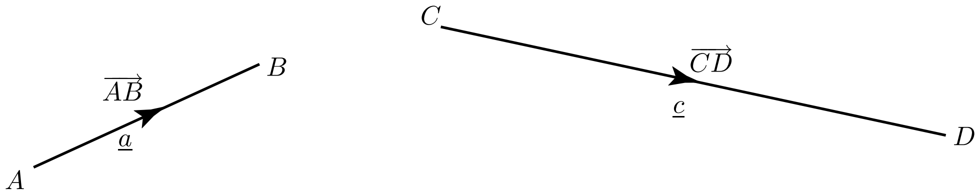
Vectors

Introduction

This leaflet explains notations in common use for describing vectors, and shows how to calculate the modulus of vectors given in cartesian form.

1. Vectors

Vectors are quantities which possess a **magnitude** and a **direction**. As such, we often represent them by **directed line segments** such as those shown below.



The arrow on the line indicates the intended direction whilst the length of the line represents the magnitude. The magnitude is also called the **modulus** or the **length** of the vector.

It is important when writing vectors that we distinguish them from scalars (or numbers) and so various notations are used to do this. We can write the vector from A to B as $\overrightarrow{AB}$. In printed work vectors are often shown with a bold typeface, as in **a**. In handwritten work we usually underline vectors, as in $\underline{a}$. Whichever way you choose it is important that vectors can be distinguished from scalars. The magnitude of a vector $\underline{a} = \overrightarrow{AB}$ is written as $|\underline{a}|$ or $|\overrightarrow{AB}|$. The magnitude is represented by the length of the directed line segment.

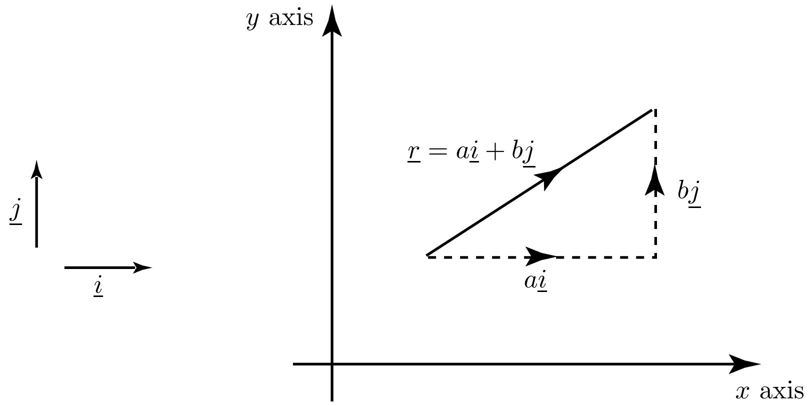
2. Unit vectors

A **unit vector** is a vector of length 1. To obtain a unit vector in the direction of any vector $\underline{a}$ we divide by its modulus. To show a vector is a unit vector we give it a 'hat', as in $\hat{\underline{a}}$.

$$\hat{\underline{a}} = \frac{\underline{a}}{|\underline{a}|}$$

3. Cartesian components

> $\underline{i}$ represents a unit vector in the direction of the positive x axis
>
> $\underline{j}$ represents a unit vector in the direction of the positive y axis



Any vector in the xy plane can be written $\underline{r} = a\underline{i} + b\underline{j}$ where a and b are numbers.

Its modulus can be found using Pythagoras' theorem:

$$|\underline{r}| = \sqrt{a^2 + b^2}$$

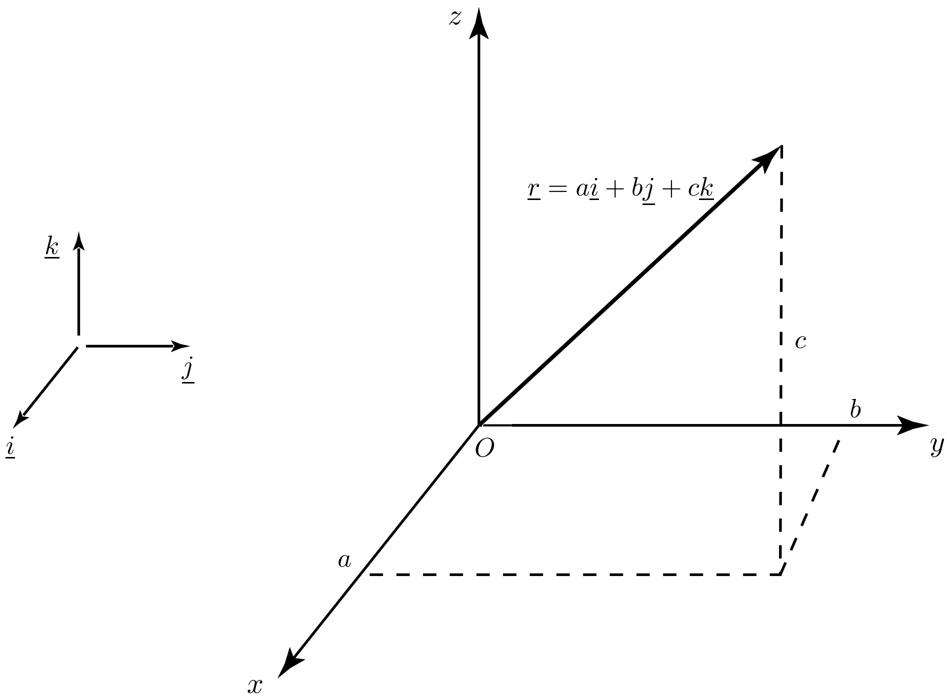
4. Three dimensions

To work in three dimensions we introduce an additional unit vector $\underline{k}$ which points in the direction of the positive z axis.

Any vector in three dimensions can be written $\underline{r} = a\underline{i} + b\underline{j} + c\underline{k}$.

Its modulus can be found using Pythagoras' theorem:

$$|\underline{r}| = \sqrt{a^2 + b^2 + c^2}$$



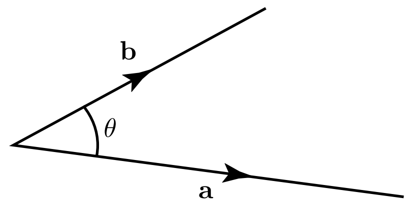
The scalar product

Introduction

In this leaflet we describe how to find the **scalar product** of two vectors.

1. Definition of the scalar product

Consider the two vectors **a** and **b** shown below. Note that the tails of the two vectors coincide and that the angle between the vectors has been labelled θ.



Their scalar product, denoted $\mathbf{a}\cdot\mathbf{b}$, is defined as $|\mathbf{a}|\,|\mathbf{b}|\cos\theta$. It is very important to use the dot in the formula. The dot is the symbol for the scalar product, and is the reason why the scalar product is also known as the **dot product**. You should never use a $\times$ sign in this context because this symbol is reserved for a quantity called the **vector product** which is quite different.

$$\text{scalar product :}\qquad \mathbf{a}\cdot\mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\cos\theta$$

Example
Vectors **a** and **b** are shown in the figure above. Suppose the vector **a** has modulus 8 and the vector **b** has modulus 7. Suppose also that the angle, θ, between these vectors is 30°. Calculate $\mathbf{a}\cdot\mathbf{b}$.

Solution

$$\begin{aligned}\mathbf{a}\cdot\mathbf{b} &= |\mathbf{a}|\,|\mathbf{b}|\cos\theta\\ &= (8)(7)\cos 30^\circ\\ &= 48.5\end{aligned}$$

The scalar product of **a** and **b** is equal to 48.5. Note that when finding a scalar product the result is always a scalar, that is a number, and not a vector.

2. A formula for finding the scalar product

A simple formula exists for finding a scalar product when the vectors are given in cartesian form.

$$\text{if} \qquad \mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} \quad \text{and} \quad \mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k} \qquad \text{then}$$
$$\mathbf{a}\cdot\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$$

Example
If $\mathbf{a} = 5\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$ and $\mathbf{b} = 8\mathbf{i} - 9\mathbf{j} + 11\mathbf{k}$, find $\mathbf{a}\cdot\mathbf{b}$.

Solution
Respective components are multiplied together and the results are added.

$$\mathbf{a}\cdot\mathbf{b} = (5)(8) + (3)(-9) + (-2)(11) = 40 - 27 - 22 = -9$$

Note again that the result is a scalar not a vector. The answer cannot contain $\mathbf{i}$, $\mathbf{j}$, or $\mathbf{k}$.

Exercises
1. If $\mathbf{a} = 2\mathbf{i} + \mathbf{j} + 3\mathbf{k}$, $\mathbf{b} = 7\mathbf{i} + \mathbf{j} + 2\mathbf{k}$ and $\mathbf{c} = -\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}$ show $\mathbf{a}\cdot\mathbf{b} = 21$, $\mathbf{b}\cdot\mathbf{c} = 1$ and $\mathbf{a}\cdot\mathbf{c} = 8$.

3. Using the scalar product to find the angle between two vectors

The scalar product is useful when you need to calculate the angle between two vectors.

Example
Find the angle between the vectors $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j} + 5\mathbf{k}$ and $\mathbf{b} = \mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$.

Solution
Their scalar product is easily shown to be 11. The modulus of $\mathbf{a}$ is $\sqrt{2^2 + 3^2 + 5^2} = \sqrt{38}$. The modulus of $\mathbf{b}$ is $\sqrt{1^2 + (-2)^2 + 3^2} = \sqrt{14}$. Using the formula for the scalar product we find

$$\begin{aligned} \mathbf{a}\cdot\mathbf{b} &= |\mathbf{a}|\,|\mathbf{b}|\,\cos\theta \\ 11 &= \sqrt{38}\,\sqrt{14}\cos\theta \end{aligned}$$

from which

$$\cos\theta = \frac{11}{\sqrt{38}\sqrt{14}} = 0.4769 \qquad \text{so that} \qquad \theta = \cos^{-1}(0.4769) = 61.5^\circ$$

In general, the angle between two vectors can be found from the following formula:

$$\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|}$$

Exercise
1. Show that the angle between the vectors $\mathbf{a} = 5\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$ and $\mathbf{b} = 8\mathbf{i} - 9\mathbf{j} + 11\mathbf{k}$ is 95.14°.

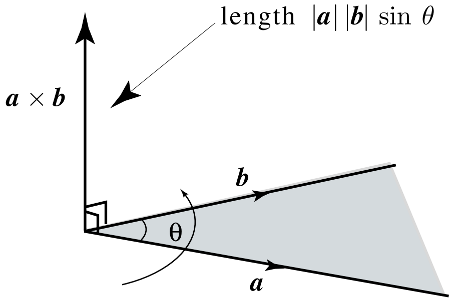
The vector product

Introduction

In this leaflet we describe how to find the **vector product** of two vectors.

1. Definition of the vector product

The result of finding the vector product of two vectors, $\mathbf{a}$ and $\mathbf{b}$, is a vector of modulus $|\mathbf{a}|\,|\mathbf{b}|\sin\theta$ in the direction of $\hat{\mathbf{e}}$, where $\hat{\mathbf{e}}$ is a unit vector perpendicular to the plane containing $\mathbf{a}$ and $\mathbf{b}$ in a sense defined by the right-handed screw rule as shown below. The symbol used for the vector product is the times sign, $\times$. Do not use a dot, $\cdot$, because this is the symbol used for a scalar product.



vector product: $\mathbf{a}\times\mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\,\sin\theta\,\hat{\mathbf{e}}$

2. A formula for finding the vector product

A formula exists for finding the vector product of two vectors given in cartesian form:

If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ then

$$\mathbf{a}\times\mathbf{b} = (a_2b_3 - a_3b_2)\mathbf{i} - (a_1b_3 - a_3b_1)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}$$

Example
Evaluate the vector product $\mathbf{a}\times\mathbf{b}$ if $\mathbf{a} = 3\mathbf{i} - 2\mathbf{j} + 5\mathbf{k}$ and $\mathbf{b} = 7\mathbf{i} + 4\mathbf{j} - 8\mathbf{k}$.

Solution
By inspection $a_1 = 3$, $a_2 = -2$, $a_3 = 5$, $b_1 = 7$, $b_2 = 4$, $b_3 = -8$, and so

$$\begin{aligned}\mathbf{a}\times\mathbf{b} &= ((-2)(-8) - (5)(4))\mathbf{i} - ((3)(-8) - (5)(7))\mathbf{j} + ((3)(4) - (-2)(7))\mathbf{k} \\ &= -4\mathbf{i} + 59\mathbf{j} + 26\mathbf{k}\end{aligned}$$

3. Using determinants to evaluate a vector product

Evaluation of a vector product using the previous formula is very cumbersome. There is a more convenient and easily remembered method for those of you who are familiar with determinants. The vector product of two vectors $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ can be found by evaluating the determinant:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

To find the $\mathbf{i}$ component of the vector product, imagine crossing out the row and column containing $\mathbf{i}$ and finding the determinant of what is left, that is

$$\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} = a_2 b_3 - a_3 b_2$$

The resulting number is the $\mathbf{i}$ component of the vector product. The $\mathbf{j}$ component is found by crossing out the row and column containing $\mathbf{j}$ and evaluating

$$\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} = a_1 b_3 - a_3 b_1$$

and then changing the sign of the result. Finally the $\mathbf{k}$ component is found by crossing out the row and column containing $\mathbf{k}$ and evaluating

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1$$

If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ then

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2 b_3 - a_3 b_2)\mathbf{i} - (a_1 b_3 - a_3 b_1)\mathbf{j} + (a_1 b_2 - a_2 b_1)\mathbf{k}$$

Example
Find the vector product of $\mathbf{a} = 3\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$ and $\mathbf{b} = 9\mathbf{i} - 6\mathbf{j} + 2\mathbf{k}$.

Solution
The two given vectors are represented in the determinant

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -4 & 2 \\ 9 & -6 & 2 \end{vmatrix}$$

Evaluating this determinant we obtain

$$\mathbf{a} \times \mathbf{b} = (-8 - (-12))\mathbf{i} - (6 - 18)\mathbf{j} + (-18 - (-36))\mathbf{k} = 4\mathbf{i} + 12\mathbf{j} + 18\mathbf{k}$$

Exercises
1. If $\mathbf{a} = 8\mathbf{i} + \mathbf{j} - 2\mathbf{k}$ and $\mathbf{b} = 5\mathbf{i} - 3\mathbf{j} + \mathbf{k}$ show that $\mathbf{a} \times \mathbf{b} = -5\mathbf{i} - 18\mathbf{j} - 29\mathbf{k}$. Show also that $\mathbf{b} \times \mathbf{a}$ is not equal to $\mathbf{a} \times \mathbf{b}$, but rather that $\mathbf{b} \times \mathbf{a} = 5\mathbf{i} + 18\mathbf{j} + 29\mathbf{k}$.

What is a complex number?

Introduction

This leaflet explains how the set of real numbers with which you are already familiar is enlarged to include further numbers called **imaginary numbers**. This leads to a study of **complex numbers** which are useful in a variety of engineering applications, especially alternating current circuit analysis.

1. Finding the square root of a negative number

It is impossible to find the square root of a negative number such as -16. If you try to find this on your calculator you will probably obtain an error message. Nevertheless it becomes useful to construct a way in which we can write down square roots of negative numbers.

We start by introducing a symbol to stand for the square root of -1. Conventionally this symbol is j. That is $j = \sqrt{-1}$. It follows that $j^2 = -1$. Using real numbers we cannot find the square root of a negative number, and so the quantity j is not real. We say it is **imaginary**.

> j is an imaginary number such that $j^2 = -1$

Even though j is not real, using it we can formally write down the square roots of any negative number as shown in the following example.

Example
Write down expressions for the square roots of a) 9, b) -9.

Solution
a) $\sqrt{9} = \pm 3$.

b) Noting that $-9 = 9 \times -1$ we can write

$$\begin{aligned} \sqrt{-9} &= \sqrt{9 \times -1} \\ &= \sqrt{9} \times \sqrt{-1} \\ &= \pm 3 \times \sqrt{-1} \end{aligned}$$

Then using the fact that $\sqrt{-1} = j$ we have

$$\sqrt{-9} = \pm 3j$$

Example
Use the fact that $j^2 = -1$ to simplify a) j^3, b) j^4.

Solution
a) $j^3 = j^2 \times j$. But $j^2 = -1$ and so $j^3 = -1 \times j = -j$.

b) $j^4 = j^2 \times j^2 = (-1) \times (-1) = 1$.

Using the imaginary number j it is possible to solve all quadratic equations.

Example
Use the formula for solving a quadratic equation to solve $2x^2 + x + 1 = 0$.

Solution
We use the formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. With $a = 2$, $b = 1$ and $c = 1$ we find

$$\begin{aligned} x &= \frac{-1 \pm \sqrt{1^2 - (4)(2)(1)}}{2(2)} \\ &= \frac{-1 \pm \sqrt{-7}}{4} \\ &= \frac{-1 \pm \sqrt{7}j}{4} \\ &= -\frac{1}{4} \pm \frac{\sqrt{7}}{4}j \end{aligned}$$

Exercises

1. Simplify a) $-j^2$, b) $(-j)^2$, c) $(-j)^3$, d) $-j^3$.
2. Solve the quadratic equation $3x^2 + 5x + 3 = 0$.

Answers
1. a) 1, b) -1, c) j, d) j. 2. $-\frac{5}{6} \pm \frac{\sqrt{11}}{6}j$.

2. Complex numbers

In the previous example we found that the solutions of $2x^2+x+1 = 0$ were $-\frac{1}{4} \pm \frac{\sqrt{7}}{4}j$. These are **complex numbers**. A complex number such as $-\frac{1}{4} + \frac{\sqrt{7}}{4}j$ is made up of two parts, a **real part**, $-\frac{1}{4}$, and an **imaginary part**, $\frac{\sqrt{7}}{4}$. We often use the letter z to stand for a complex number and write $z = a + bj$, where a is the real part and b is the imaginary part.

$$z = a + bj$$

where a is the real part and b is the imaginary part of the complex number.

Exercises
1. State the real and imaginary parts of: a) $13 - 5j$, b) $1 - 0.35j$, c) $\cos\theta + j\sin\theta$.

Answers
1. a) real part 13, imaginary part -5, b) $1, -0.35$, c) $\cos\theta$, $\sin\theta$.

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7.2

Complex arithmetic

Introduction.

This leaflet describes how complex numbers are added, subtracted, multiplied and divided.

1. Addition and subtraction of complex numbers

Given two complex numbers we can find their sum and difference in an obvious way.

If $z_1 = a_1 + b_1 j$ and $z_2 = a_2 + b_2 j$ then

$$z_1 + z_2 = (a_1 + a_2) + (b_1 + b_2)j$$

$$z_1 - z_2 = (a_1 - a_2) + (b_1 - b_2)j$$

So, to add the complex numbers we simply add the real parts together and add the imaginary parts together.

Example
If $z_1 = 13 + 5j$ and $z_2 = 8 - 2j$ find a) $z_1 + z_2$, b) $z_2 - z_1$.

Solution
a) $z_1 + z_2 = (13 + 5j) + (8 - 2j) = 21 + 3j$.

b) $z_2 - z_1 = (8 - 2j) - (13 + 5j) = -5 - 7j$

2. Multiplication of complex numbers

To multiply two complex numbers we use the normal rules of algebra and also the fact that $j^2 = -1$. If z_1 and z_2 are the two complex numbers their product is written $z_1 z_2$.

Example
If $z_1 = 5 - 2j$ and $z_2 = 2 + 4j$ find $z_1 z_2$.

Solution

$$z_1 z_2 = (5 - 2j)(2 + 4j) = 10 + 20j - 4j - 8j^2$$

Replacing j^2 by -1 we obtain

$$z_1 z_2 = 10 + 16j - 8(-1) = 18 + 16j$$

In general we have the following result:

If $z_1 = a_1 + b_1 j$ and $z_2 = a_2 + b_2 j$ then

$$\begin{aligned} z_1 z_2 = (a_1 + b_1 j)(a_2 + b_2 j) &= a_1 a_2 + a_1 b_2 j + b_1 a_2 j + b_1 b_2 j^2 \\ &= (a_1 a_2 - b_1 b_2) + j(a_1 b_2 + a_2 b_1) \end{aligned}$$

3. Division of complex numbers

To divide complex numbers we need to make use of the **complex conjugate**. Given a complex number, z, its conjugate, written $\bar{z}$, is found by changing the sign of the imaginary part. For example, the complex conjugate of $z = 3 + 2j$ is $\bar{z} = 3 - 2j$. Division is illustrated in the following example.

Example
Find $\dfrac{z_1}{z_2}$ when $z_1 = 3 + 2j$ and $z_2 = 4 - 3j$.

Solution
We require

$$\frac{z_1}{z_2} = \frac{3 + 2j}{4 - 3j}$$

Both numerator and denominator are multiplied by the complex conjugate of the denominator. Overall, this is equivalent to multiplying by 1 and so the fraction remains unaltered, but it will have the effect of making the denominator purely real, as you will see.

$$\begin{aligned} \frac{3+2j}{4-3j} &= \frac{3+2j}{4-3j} \times \frac{4+3j}{4+3j} \\ &= \frac{(3+2j)(4+3j)}{(4-3j)(4+3j)} \\ &= \frac{12 + 9j + 8j + 6j^2}{16 + 12j - 12j - 9j^2} \\ &= \frac{6 + 17j}{25} \qquad \text{(the denominator is now seen to be real)} \\ &= \frac{6}{25} + \frac{17}{25} j \end{aligned}$$

Exercises

1. If $z_1 = 1 + j$ and $z_2 = 3 + 2j$ find a) $z_1 z_2$, b) $\overline{z_1}$, c) $\overline{z_2}$, d) $z_1 \overline{z_1}$, e) $z_2 \overline{z_2}$.
2. If $z_1 = 1 + j$ and $z_2 = 3 + 2j$ find a) $\frac{z_1}{z_2}$, b) $\frac{z_2}{z_1}$, c) $z_1/\overline{z_1}$, d) $z_2/\overline{z_2}$.
3. Find a) $\frac{7-6j}{2j}$, b) $\frac{3+9j}{1-2j}$, c) $\frac{1}{j}$.

Answers

1. a) $1 + 5j$, b) $1 - j$, c) $3 - 2j$, d) 2, e) 13.
2. a) $\frac{5}{13} + \frac{j}{13}$, b) $\frac{5}{2} - \frac{j}{2}$, c) j, d) $\frac{5}{13} + \frac{12}{13} j$.
3. a) $-3 - \frac{7}{2} j$, b) $-3 + 3j$, c) $-j$.

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7.3

The Argand diagram

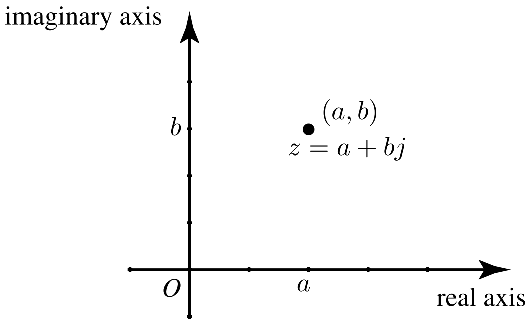
Introduction

Engineers often find a pictorial representation of complex numbers useful.

Such a representation is known as an **Argand diagram**. This leaflet explains how to draw an Argand diagram.

1. The Argand diagram

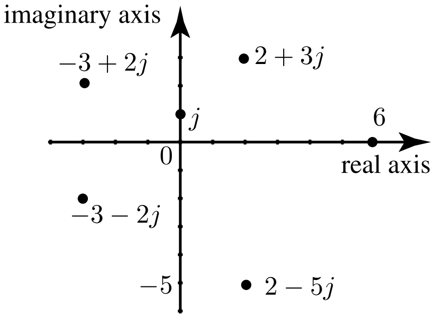
The complex number $z = a + bj$ is plotted as a point with coordinates (a, b) as shown. Because the real part of z is plotted on the horizontal axis we often refer to this as the **real axis**. The imaginary part of z is plotted on the vertical axis and so we refer to this as the **imaginary axis**. Such a diagram is called an **Argand diagram**.



The complex number $z = a + bj$ is plotted as the point with coordinates (a, b).

Example
Plot the complex numbers $2 + 3j$, $-3 + 2j$, $-3 - 2j$, $2 - 5j$, 6, j on an Argand diagram.

Solution
The figure below shows the Argand diagram. Note that purely real numbers lie on the real axis. Purely imaginary numbers lie on the imaginary axis. Note that complex conjugate pairs such as $-3 \pm 2j$ lie symmetrically on opposite sides of the real axis.



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7.4

The polar form

Introduction

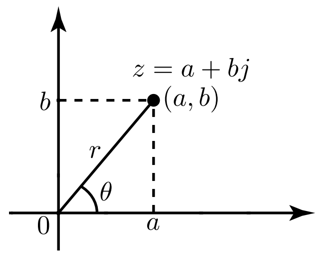
From an Argand diagram the **modulus** and the **argument** of a complex number can be defined. These provide an alternative way of describing complex numbers known as the **polar form**. This leaflet explains how to find the modulus and argument.

1. The modulus and argument of a complex number

The Argand diagram below shows the complex number $z = a + bj$. The distance of the point (a, b) from the origin is called the **modulus**, or **magnitude**, of the complex number and has the symbol r. Alternatively, r is written as $|z|$. The modulus is never negative. The modulus can be found using Pythagoras' theorem, that is

$$|z| = r = \sqrt{a^2 + b^2}$$

The angle between the positive x axis and a line joining (a, b) to the origin is called the **argument** of the complex number. It is abbreviated to $\arg(z)$ and has been given the symbol θ.



We usually measure θ so that it lies between $-\pi$ and π (that is, between $-180°$ and $180°$). Angles measured anticlockwise from the positive x axis are conventionally positive, whereas angles measured clockwise are negative. Knowing values for a and b, trigonometry can be used to determine θ. Specifically,

$$\tan\theta = \frac{b}{a} \qquad \text{so that} \qquad \theta = \tan^{-1}\left(\frac{b}{a}\right)$$

but care must be taken when using a calculator to find an inverse tangent that the solution obtained is in the correct quadrant. Drawing an Argand diagram will always help to identify the correct quadrant. The position of a complex number is uniquely determined by giving its modulus and argument. This description is known as the **polar form**. When the modulus and argument of a complex number, z, are known we write the complex number as $z = r\angle\theta$.

Polar form of a complex number with modulus r and argument θ:

$$z = r\angle\theta$$

Example

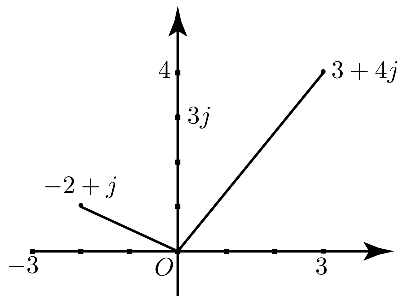
Plot the following complex numbers on an Argand diagram and find their moduli.

a) $z_1 = 3 + 4j$, b) $z_2 = -2 + j$, c) $z_3 = 3j$.

Solution

The complex numbers are shown in the figure below. In each case we can use Pythagoras' theorem to find the modulus.

a) $|z_1| = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$, b) $|z_2| = \sqrt{(-2)^2 + 1^2} = \sqrt{5}$ or 2.236, c) $|z_3| = \sqrt{3^2 + 0^2} = 3$.



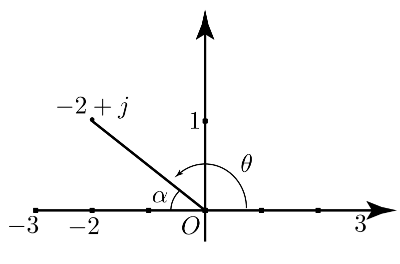
Example

Find the arguments of the complex numbers in the previous example.

Solution

a) $z_1 = 3 + 4j$ is in the first quadrant. Its argument is given by $\theta = \tan^{-1}\frac{4}{3}$. Using a calculator we find $\theta = 0.927$ radians, or 53.13°.

b) $z_2 = -2 + j$ is in the second quadrant. To find its argument we seek an angle, θ, in the second quadrant such that $\tan\theta = \frac{1}{-2}$. To calculate this correctly it may help to refer to the figure below in which α is an acute angle with $\tan\alpha = \frac{1}{2}$. From a calculator $\alpha = 0.464$ and so $\theta = \pi - 0.464 = 2.678$ radians. In degrees, $\alpha = 26.57°$ so that $\theta = 180° - 26.57° = 153.43°$.



c) $z_3 = 3j$ is purely imaginary. Its argument is $\frac{\pi}{2}$, or 90°.

Exercises

1. Plot the following complex numbers on an Argand diagram and find their moduli and arguments.

a) $z = 9$, b) $z = -5$, c) $z = 1 + 2j$, d) $z = -1 - j$, e) $z = 8j$, f) $-5j$.

Answers

1. a) $|z| = 9$, $\arg(z) = 0$, b) $|z| = 5$, $\arg(z) = \pi$, or 180°, c) $|z| = \sqrt{5}$, $\arg(z) = 1.107$ or 63.43°, d) $|z| = \sqrt{2}$, $\arg(z) = -\frac{3\pi}{4}$ or −135°, e) $|z| = 8$, $\arg(z) = \frac{\pi}{2}$ or 90°, f) $|z| = 5$, $\arg(z) = -\frac{\pi}{2}$ or −90°.

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7.5

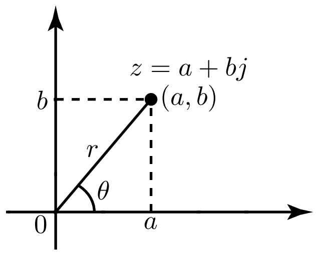
The form $r(\cos\theta + j\sin\theta)$

Introduction

Any complex number can be written in the form $z = r(\cos\theta + j\sin\theta)$ where r is its modulus and θ is its argument. This leaflet explains this form.

1. The form $r(\cos\theta + j\sin\theta)$

Consider the figure below which shows the complex number $z = a + bj = r\angle\theta$.



Using trigonometry we can write

$$\cos\theta = \frac{a}{r} \qquad \text{and} \qquad \sin\theta = \frac{b}{r}$$

so that, by rearranging,

$$a = r\cos\theta \qquad \text{and} \qquad b = r\sin\theta$$

We can use these results to find the real and imaginary parts of a complex number given in polar form:

> if $z = r\angle\theta$, the real and imaginary parts of z are:
>
> $$a = r\cos\theta \qquad \text{and} \qquad b = r\sin\theta, \qquad \text{respectively}$$

Using these results we can then write $z = a + bj$ as

$$\begin{aligned} z = a + bj &= r\cos\theta + jr\sin\theta \\ &= r(\cos\theta + j\sin\theta) \end{aligned}$$

This is an alternative way of expressing the complex number with modulus r and argument θ.

$$z = a + bj = r\angle\theta = r(\cos\theta + j\sin\theta)$$

Example

State the modulus and argument of a) $z = 9(\cos 40° + j\sin 40°)$, b) $z = 17(\cos 3.2 + j\sin 3.2)$.

Solution

a) Comparing the given complex number with the standard form $r(\cos\theta + j\sin\theta)$ we see that $r = 9$ and $\theta = 40°$. The modulus is 9 and the argument is 40°.

b) Comparing the given complex number with the standard form $r(\cos\theta + j\sin\theta)$ we see that $r = 17$ and $\theta = 3.2$ radians. The modulus is 17 and the argument is 3.2 radians.

Example

a) Find the modulus and argument of the complex number $z = 5j$.

b) Express $5j$ in the form $r(\cos\theta + j\sin\theta)$.

Solution

a) On an Argand diagram the complex number $5j$ lies on the positive vertical axis a distance 5 from the origin. Thus $5j$ is a complex number with modulus 5 and argument $\frac{\pi}{2}$.

b)

$$z = 5j = 5(\cos\frac{\pi}{2} + j\sin\frac{\pi}{2})$$

Using degrees we would write

$$z = 5j = 5(\cos 90° + j\sin 90°)$$

Example

a) State the modulus and argument of the complex number $z = 4\angle(\pi/3)$.

b) Express $z = 4\angle(\pi/3)$ in the form $r(\cos\theta + j\sin\theta)$.

Solution

a) Its modulus is 4 and its argument is $\frac{\pi}{3}$.

b) $z = 4(\cos\frac{\pi}{3} + j\sin\frac{\pi}{3})$.

Noting $\cos\frac{\pi}{3} = \frac{1}{2}$ and $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$ the complex number can be written $2 + 2\sqrt{3}j$.

Exercises

1. By first finding the modulus and argument express $z = 3j$ in the form $r(\cos\theta + j\sin\theta)$.

2. By first finding the modulus and argument express $z = -3$ in the form $r(\cos\theta + j\sin\theta)$.

3. By first finding the modulus and argument express $z = -1 - j$ in the form $r(\cos\theta + j\sin\theta)$.

Answers

1. $3(\cos\frac{\pi}{2} + j\sin\frac{\pi}{2})$. 2. $3(\cos\pi + j\sin\pi)$.

3. $\sqrt{2}(\cos(-135°) + j\sin(-135°)) = \sqrt{2}(\cos 135° - j\sin 135°)$.

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7.6

Multiplication and division in polar form

Introduction

When two complex numbers are given in polar form it is particularly simple to multiply and divide them. This is an advantage of using the polar form.

1. Multiplication and division of complex numbers in polar form

If $z_1 = r_1\angle\theta_1$ and $z_2 = r_2\angle\theta_2$ then

$$z_1z_2 = r_1r_2\angle(\theta_1+\theta_2), \qquad \frac{z_1}{z_2} = \frac{r_1}{r_2}\angle(\theta_1-\theta_2)$$

Note that to multiply the two numbers we multiply their moduli and add their arguments.

To divide, we divide their moduli and subtract their arguments.

Example
If $z_1 = 5\angle(\pi/6)$, and $z_2 = 4\angle(-\pi/4)$ find a) z_1z_2, b) $\frac{z_1}{z_2}$, c) $\frac{z_2}{z_1}$.

Solution
a) To multiply the two complex numbers we multiply their moduli and add their arguments. Therefore

$$z_1z_2 = 20\angle\left(\frac{\pi}{6}+\left(-\frac{\pi}{4}\right)\right) = 20\angle\left(-\frac{\pi}{12}\right)$$

b) To divide the two complex numbers we divide their moduli and subtract their arguments.

$$\frac{z_1}{z_2} = \frac{5}{4}\angle\left(\frac{\pi}{6}-\left(-\frac{\pi}{4}\right)\right) = \frac{5}{4}\angle\frac{5\pi}{12}$$

c)

$$\frac{z_2}{z_1} = \frac{4}{5}\angle\left(-\frac{\pi}{4}-\frac{\pi}{6}\right) = \frac{4}{5}\angle\left(-\frac{5\pi}{12}\right)$$

Exercises

1. If $z_1 = 7\angle\frac{\pi}{3}$ and $z_2 = 6\angle\frac{\pi}{2}$ find a) z_1z_2, b) $\frac{z_1}{z_2}$, c) $\frac{z_2}{z_1}$, d) z_1^2, e) z_2^3.

Answers
1. a) $42\angle\frac{5\pi}{6}$, b) $\frac{7}{6}\angle-\frac{\pi}{6}$, c) $\frac{6}{7}\angle\frac{\pi}{6}$, d) $49\angle\frac{2\pi}{3}$, e) $216\angle\frac{3\pi}{2}$.

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7.7

The exponential form

Introduction

In addition to the cartesian and polar forms of a complex number there is a third form in which a complex number may be written – the **exponential form**. In this leaflet we explain this form.

1. Euler's relations

Two important results in complex number theory are known as **Euler's relations**. These link the exponential function and the trigonometric functions. They state:

Euler's relations:

$$e^{j\theta} = \cos\theta + j\sin\theta, \qquad e^{-j\theta} = \cos\theta - j\sin\theta$$

The derivation of these relations is beyond the scope of this leaflet. By firstly adding, and then subtracting, Euler's relations we can obtain expressions for the trigonometric functions in terms of exponential functions. Try this!

$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}, \qquad \sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

2. The exponential form of a complex number

Using the polar form, a complex number with modulus r and argument θ may be written

$$z = r(\cos\theta + j\sin\theta)$$

It follows immediately from Euler's relations that we can also write this complex number in **exponential form** as $z = re^{j\theta}$.

Exponential form

$$z = r\,e^{j\theta}$$

When using this form you should ensure that all angles are measured in radians and not degrees.

Example

State the modulus and argument of the following complex numbers:

a) $z = 5e^{j\pi/6}$, b) $z = 0.01e^{0.02j}$, c) $3e^{-j\pi/2}$, d) $5e^2$.

Solution
In each case compare the given number with the standard form $z = re^{j\theta}$ to identify the modulus r and the argument θ.

a) The modulus and argument of $5e^{j\pi/6}$ are 5 and $\frac{\pi}{6}$ respectively.

b) The modulus and argument of $0.01e^{0.02j}$ are 0.01 and 0.02 respectively.

c) The modulus and argument of $3e^{-j\pi/2}$ are 3 and $-\frac{\pi}{2}$ respectively.

d) The number $5e^2$ is purely real, and can be evaluated using a calculator. Its modulus is 36.95 and its argument is zero.

Example
Find the real and imaginary parts of $z = 5e^{2j}$.

Solution
Recall that $e^{j\theta} = \cos\theta + j\sin\theta$. Then

$$\begin{aligned} 5e^{2j} &= 5(\cos 2 + j\sin 2) \\ &= 5\cos 2 + (5\sin 2)j \end{aligned}$$

The real part is $5\cos 2$ which equals -2.08. The imaginary part is $5\sin 2$, that is 4.55 (to 2dp).

Example
Express the number $z = 3 + 3j$ in exponential form.

Solution
To express a number in exponential form we must first find its modulus and argument. The modulus of $3 + 3j$ is $\sqrt{3^2 + 3^2} = \sqrt{18}$. The complex number lies in the first quadrant of the Argand diagram and so its argument θ is given by $\theta = \tan^{-1}\frac{3}{3} = \frac{\pi}{4}$. Thus

$$z = 3 + 3j = \sqrt{18}e^{j\pi/4}$$

Exercises
1. State the modulus and argument of each of the following complex numbers:

a) $5e^{0.3j}$, b) $4e^{-j2\pi/3}$, c) $e^{2\pi j}$, d) $0.35e^{-0.2j}$.

2. Express each of the following in the form $re^{j\theta}$.

a) $3\angle(\pi/3)$, b) $\sqrt{2}\angle(\pi/4)$, c) $3\angle(-\pi/4)$, d) $5\angle 0$, e) $17\angle(\pi/2)$.

3. Express each of the following in the form $a + bj$.

a) $13e^{j\pi/3}$, b) $13e^{-j\pi/3}$, c) $4e^{2\pi j}$, d) $7e^{0.2j}$.

4. Show that e^{1+3j} is equal to $e^1 e^{3j}$. Hence deduce $e^{1+3j} = -2.69 + 0.38j$.

Answers
1. a) 5, 0.3 radians, b) 4, $-2\pi/3$ radians, c) $1, 2\pi$ radians, d) 0.35, -0.2 radians.

2. a) $3e^{j\pi/3}$, b) $\sqrt{2}e^{j\pi/4}$, c) $3e^{-j\pi/4}$, d) $5e^0 = 5$, e) $17e^{j\pi/2}$.

3. a) $6.5 + 11.3j$, b) $6.5 - 11.3j$, c) 4, d) $6.86 + 1.39j$.

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8.1

Introduction to differentiation

Introduction

This leaflet provides a rough and ready introduction to **differentiation**. This is a technique used to calculate the gradient, or slope, of a graph at different points.

1. The gradient function

Given a function, for example, $y = x^2$, it is possible to derive a formula for the gradient of its graph. We can think of this formula as the **gradient function**, precisely because it tells us the gradient of the graph. For example,

$$\text{when } y = x^2 \quad \text{the gradient function is} \quad 2x$$

So, the gradient of the graph of $y = x^2$ at any point is twice the x value there. To understand how this formula is actually found you would need to refer to a textbook on calculus. The important point is that using this formula we can calculate the gradient of $y = x^2$ at different points on the graph. For example,

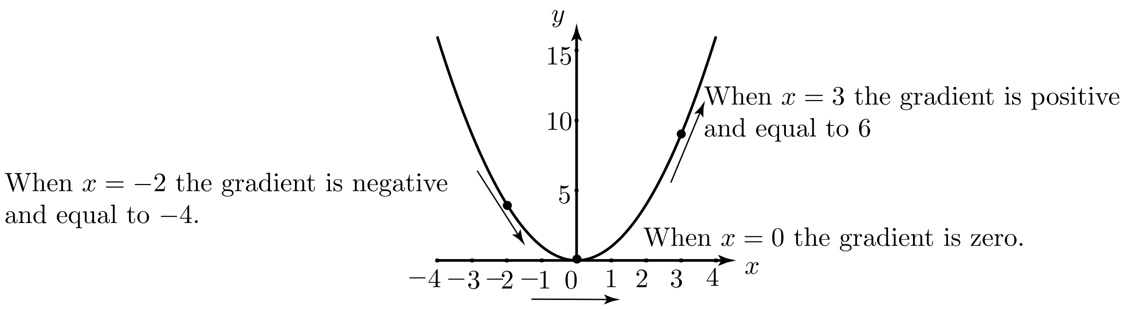
$$\text{when } x = 3, \text{ the gradient is } 2 \times 3 = 6.$$

$$\text{when } x = -2, \text{ the gradient is } 2 \times (-2) = -4.$$

How do we interpret these numbers? A gradient of 6 means that values of y are increasing at the rate of 6 units for every 1 unit increase in x. A gradient of -4 means that values of y are decreasing at a rate of 4 units for every 1 unit increase in x.

Note that when $x = 0$, the gradient is $2 \times 0 = 0$.

Below is a graph of the function $y = x^2$. Study the graph and you will note that when $x = 3$ the graph has a positive gradient. When $x = -2$ the graph has a negative gradient. When $x = 0$ the gradient of the graph is zero. Note how these properties of the graph can be predicted from knowledge of the gradient function, $2x$.



Example
When $y = x^3$, its gradient function is $3x^2$. Calculate the gradient of the graph of $y = x^3$ when
a) $x = 2$, b) $x = -1$, c) $x = 0$.

Solution
a) When $x = 2$ the gradient function is $3(2)^2 = 12$.

b) When $x = -1$ the gradient function is $3(-1)^2 = 3$.

c) When $x = 0$ the gradient function is $3(0)^2 = 0$.

2. Notation for the gradient function

You will need to use a notation for the gradient function which is in widespread use.

> If y is a function of x, that is $y = f(x)$, we write its gradient function as $\frac{\mathrm{d}y}{\mathrm{d}x}$.

$\frac{\mathrm{d}y}{\mathrm{d}x}$, pronounced 'dee y by dee x', is not a fraction even though it might look like one! This notation can be confusing. Think of $\frac{\mathrm{d}y}{\mathrm{d}x}$ as the 'symbol' for the gradient function of $y = f(x)$. The process of finding $\frac{\mathrm{d}y}{\mathrm{d}x}$ is called **differentiation with respect to** x.

Example
For any value of n, the gradient function of x^n is nx^{n-1}. We write:

$$\text{if} \qquad y = x^n, \qquad \text{then} \qquad \frac{\mathrm{d}y}{\mathrm{d}x} = nx^{n-1}$$

You have seen specific cases of this result earlier on. For example, if $y = x^3$, $\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2$.

3. More notation and terminology

When $y = f(x)$ alternative ways of writing the gradient function, $\frac{\mathrm{d}y}{\mathrm{d}x}$, are y', pronounced 'y dash', or $\frac{\mathrm{d}f}{\mathrm{d}x}$, or f', pronounced 'f dash'. In practice you do not need to remember the formulas for the gradient functions of all the common functions. Engineers usually refer to a table known as a *Table of Derivatives*. A **derivative** is another name for a gradient function. Such a table is available in leaflet 8.2. The derivative is also known as the **rate of change** of a function.

Exercises
1. Given that when $y = x^2$, $\frac{\mathrm{d}y}{\mathrm{d}x} = 2x$, find the gradient of $y = x^2$ when $x = 7$.

2. Given that when $y = x^n$, $\frac{\mathrm{d}y}{\mathrm{d}x} = nx^{n-1}$, find the gradient of $y = x^4$ when a) $x = 2$, b) $x = -1$.

3. Find the rate of change of $y = x^3$ when a) $x = -2$, b) $x = 6$.

4. Given that when $y = 7x^2 + 5x$, $\frac{\mathrm{d}y}{\mathrm{d}x} = 14x + 5$, find the gradient of $y = 7x^2 + 5x$ when $x = 2$.

Answers
1. 14. 2. a) 32, b) −4. 3. a) 12, b) 108. 4. 33.

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8.2

Table of derivatives

Introduction

This leaflet provides a table of common functions and their derivatives.

1. The table of derivatives

$y = f(x)$	$\frac{dy}{dx} = f'(x)$
k, any constant	0
x	1
x^2	$2x$
x^3	$3x^2$
x^n, any constant n	nx^{n-1}
e^x	e^x
e^{kx}	ke^{kx}
$\ln x = \log_e x$	$\frac{1}{x}$
$\sin x$	$\cos x$
$\sin kx$	$k \cos kx$
$\cos x$	$-\sin x$
$\cos kx$	$-k \sin kx$
$\tan x = \frac{\sin x}{\cos x}$	$\sec^2 x$
$\tan kx$	$k \sec^2 kx$
$\text{cosec}\, x = \frac{1}{\sin x}$	$-\text{cosec}\, x \cot x$
$\sec x = \frac{1}{\cos x}$	$\sec x \tan x$
$\cot x = \frac{\cos x}{\sin x}$	$-\text{cosec}^2 x$
$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}$
$\cos^{-1} x$	$\frac{-1}{\sqrt{1-x^2}}$
$\tan^{-1} x$	$\frac{1}{1+x^2}$
$\cosh x$	$\sinh x$
$\sinh x$	$\cosh x$
$\tanh x$	$\text{sech}^2 x$
$\text{sech}\, x$	$-\text{sech}\, x \tanh x$
$\text{cosech}\, x$	$-\text{cosech}\, x \coth x$
$\coth x$	$-\text{cosech}^2 x$
$\cosh^{-1} x$	$\frac{1}{\sqrt{x^2-1}}$
$\sinh^{-1} x$	$\frac{1}{\sqrt{x^2+1}}$
$\tanh^{-1} x$	$\frac{1}{1-x^2}$

Exercises

1. In each case, use the table of derivatives to write down $\frac{dy}{dx}$.

a) $y = 8$

b) $y = -2$

c) $y = 0$

d) $y = x$

e) $y = x^5$

f) $y = x^7$

g) $y = x^{-3}$

h) $y = x^{1/2}$

i) $y = x^{-1/2}$

j) $y = \sin x$

k) $y = \cos x$

l) $y = \sin 4x$

m) $y = \cos \frac{1}{2}x$

n) $y = e^{4x}$

o) $y = e^x$

p) $y = e^{-2x}$

q) $y = e^{-x}$

r) $y = \ln x$

s) $y = \log_e x$

t) $y = \sqrt{x}$

u) $y = \sqrt[3]{x}$

v) $y = \frac{1}{\sqrt{x}}$

w) $y = e^{x/2}$

2. You should be able to use the table when other variables are used. Find $\frac{dy}{dt}$ if

a) $y = e^{7t}$, b) $y = t^4$, c) $y = t^{-1}$, d) $y = \sin 3t$.

Answers

1. a) 0, b) 0, c) 0, d) 1, e) $5x^4$, f) $7x^6$, g) $-3x^{-4}$, h) $\frac{1}{2}x^{-1/2}$, i) $-\frac{1}{2}x^{-3/2}$, j) $\cos x$, k) $-\sin x$, l) $4\cos 4x$, m) $-\frac{1}{2}\sin\frac{1}{2}x$, n) $4e^{4x}$, o) e^x, p) $-2e^{-2x}$, q) $-e^{-x}$, r) $\frac{1}{x}$, s) $\frac{1}{x}$ t) $\frac{1}{2}x^{-1/2} = \frac{1}{2x^{1/2}} = \frac{1}{2\sqrt{x}}$, u) $\frac{1}{3}x^{-2/3} = \frac{1}{3x^{2/3}} = \frac{1}{3\sqrt[3]{x^2}}$, v) $-\frac{1}{2}x^{-3/2}$, w) $\frac{1}{2}e^{x/2}$.

2. a) $7e^{7t}$, b) $4t^3$, c) $-\frac{1}{t^2}$, d) $3\cos 3t$.

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8.3

Linearity rules

Introduction

There are two rules known as **linearity rules** which, when used with a Table of Derivatives, enable us to differentiate a wider range of functions. These rules are summarised here.

1. Some notation

Before we look at the rules, we need to be clear about the meaning of the notation $\frac{\mathrm{d}}{\mathrm{d}x}$.

When we are given a function $y(x)$ and are asked to find $\frac{\mathrm{d}y}{\mathrm{d}x}$ we are being instructed to carry out an operation on the function $y(x)$. The operation is that of differentiation. A notation for this operation is used widely:

$\frac{\mathrm{d}}{\mathrm{d}x}$ stands for the operation: 'differentiate with respect to x'

For example, $\frac{\mathrm{d}}{\mathrm{d}x}(x^3) = 3x^2$ and $\frac{\mathrm{d}}{\mathrm{d}x}(\sin x) = \cos x$.

2. Differentiation of a function multiplied by a constant

If k is a constant and f is a function of x, then

$$\frac{\mathrm{d}}{\mathrm{d}x}(kf) = k\,\frac{\mathrm{d}f}{\mathrm{d}x}$$

This means that a constant factor can be brought outside the differentiation operation.

Example

Given that $\frac{\mathrm{d}}{\mathrm{d}x}(x^3) = 3x^2$, then it follows that

$$\frac{\mathrm{d}}{\mathrm{d}x}(7x^3) = 7 \times \frac{\mathrm{d}}{\mathrm{d}x}(x^3) = 7 \times 3x^2 = 21x^2$$

Given that $\frac{\mathrm{d}}{\mathrm{d}x}(\sin x) = \cos x$, then it follows that

$$\frac{\mathrm{d}}{\mathrm{d}x}(8\sin x) = 8 \times \frac{\mathrm{d}}{\mathrm{d}x}(\sin x) = 8\cos x$$

3. Differentiation of the sum or difference of two functions

If f and g are functions of x, then

$$\frac{\mathrm{d}}{\mathrm{d}x}(f+g) = \frac{\mathrm{d}f}{\mathrm{d}x} + \frac{\mathrm{d}g}{\mathrm{d}x} \qquad \frac{\mathrm{d}}{\mathrm{d}x}(f-g) = \frac{\mathrm{d}f}{\mathrm{d}x} - \frac{\mathrm{d}g}{\mathrm{d}x}$$

This means that to differentiate a sum of two functions, simply differentiate each separately and then add the results. Similarly, to differentiate the difference of two functions, differentiate each separately and then find the difference of the results.

Example

Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ when $y = x^2 + x$.

Solution

We require $\frac{\mathrm{d}}{\mathrm{d}x}(x^2 + x)$. The sum rule tells us to differentiate each term separately. Thus

$$\frac{\mathrm{d}}{\mathrm{d}x}(x^2 + x) = \frac{\mathrm{d}}{\mathrm{d}x}(x^2) + \frac{\mathrm{d}}{\mathrm{d}x}(x) = 2x + 1$$

So $\frac{\mathrm{d}y}{\mathrm{d}x} = 2x + 1$.

Example

Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ when $y = \mathrm{e}^{2x} - \sin 3x$.

Solution

The difference rule tells us to differentiate each term separately.

$$\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^{2x} - \sin 3x) = \frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^{2x}) - \frac{\mathrm{d}}{\mathrm{d}x}(\sin 3x) = 2\mathrm{e}^{2x} - 3\cos 3x$$

So

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2\mathrm{e}^{2x} - 3\cos 3x$$

Exercises

In each case use a Table of Derivatives and the rules on this leaflet to find $\frac{\mathrm{d}y}{\mathrm{d}x}$.

1. $y = \mathrm{e}^{5x} + \cos 2x$
2. $y = x^2 - \sin x$
3. $y = 3x^2 + 7x + 2$
4. $y = 5$
5. $y = 8\mathrm{e}^{-9x}$

Answers

1. $5\mathrm{e}^{5x} - 2\sin 2x$, 2. $2x - \cos x$, 3. $6x + 7$, 4. 0, 5. $-72\mathrm{e}^{-9x}$.

Product and quotient rules

Introduction

As their names suggest, the **product rule** and the **quotient rule** are used to differentiate products of functions and quotients of functions. This leaflet explains how.

1.The product rule

It is appropriate to use this rule when you want to differentiate two functions which are multiplied together. For example

$$y = \mathrm{e}^x \sin x \qquad \text{is a product of the functions } \mathrm{e}^x \text{ and } \sin x$$

In the rule which follows we let u stand for the first of the functions and v stand for the second.

If u and v are functions of x, then

$$\frac{\mathrm{d}}{\mathrm{d}x}(uv) = u\frac{\mathrm{d}v}{\mathrm{d}x} + v\frac{\mathrm{d}u}{\mathrm{d}x}$$

Example

If $y = 7x\mathrm{e}^{2x}$ find $\frac{\mathrm{d}y}{\mathrm{d}x}$.

Solution

Comparing the given function with the product rule we let

$$u = 7x, \qquad v = \mathrm{e}^{2x}$$

It follows that

$$\frac{\mathrm{d}u}{\mathrm{d}x} = 7, \qquad \text{and} \qquad \frac{\mathrm{d}v}{\mathrm{d}x} = 2\mathrm{e}^{2x}$$

Thus, using the product rule,

$$\frac{\mathrm{d}}{\mathrm{d}x}(7x\mathrm{e}^{2x}) = 7x(2\mathrm{e}^{2x}) + \mathrm{e}^{2x}(7) = 7\mathrm{e}^{2x}(2x+1)$$

2. The quotient rule

It is appropriate to use this rule when you want to differentiate a quotient of two functions, that is, one function divided by another. For example

$$y = \frac{e^x}{\sin x} \qquad \text{is a quotient of the functions } e^x \text{ and } \sin x$$

In the rule which follows we let u stand for the function in the numerator and v stand for the function in the denominator.

If u and v are functions of x, then

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

Example

If $y = \dfrac{\sin x}{3x^2}$ find $\dfrac{dy}{dx}$.

Solution

Comparing the given function with the quotient rule we let

$$u = \sin x, \qquad \text{and} \qquad v = 3x^2$$

It follows that

$$\frac{du}{dx} = \cos x \qquad \text{and} \qquad \frac{dv}{dx} = 6x$$

Applying the quotient rule gives

$$\frac{dy}{dx} = \frac{3x^2\cos x - \sin x\,(6x)}{9x^4} = \frac{3x(x\cos x - 2\sin x)}{9x^4} = \frac{x\cos x - 2\sin x}{3x^3}$$

Exercises

Choose an appropriate rule in each case to find $\dfrac{dy}{dx}$.

1. $y = x^2 \sin x$
2. $y = e^x \cos x$
3. $y = \frac{e^x}{x^2+1}$
4. $y = \frac{x^2+1}{e^x}$
5. $y = 7x \log_e x$
6. $y = \frac{x-1}{\sin 2x}$

Answers

1. $x^2\cos x + 2x\sin x$. 2. $-e^x\sin x + e^x\cos x = e^x(\cos x - \sin x)$. 3. $\frac{e^x(x^2-2x+1)}{(x^2+1)^2}$.

4. $\frac{2x-x^2-1}{e^x}$. 5. $7(1+\log_e x)$. 6. $\frac{\sin 2x - 2(x-1)\cos 2x}{\sin^2 2x}$.

The chain rule

Introduction

The **chain rule** is used when it is necessary to differentiate a function of a function.

This rule is summarised here.

1. The chain rule

Consider the function $y = (\sin x)^3$. This process involves cubing the function $\sin x$.

Consider also the function $y = \log_e(x^3 + 5x)$. Here we are finding the logarithm of the function $x^3 + 5x$.

In both cases we are finding a **function of a function.**

The chain rule is used to differentiate such composite functions and is illustrated in the examples which follow.

Example
Find $\dfrac{dy}{dx}$ when $y = \sin(5x + 3)$.

Solution
Notice that $5x + 3$ is a function of x, so $\sin(5x + 3)$ is a function of a function.

To simplify the problem we can introduce a new variable z and write $z = 5x + 3$ so that y becomes

$$y = \sin z$$

Then, differentiating this with respect to z,

$$\frac{dy}{dz} = \cos z$$

Now, in fact, we want $\dfrac{dy}{dx}$. The chain rule states

$$\frac{dy}{dx} = \frac{dy}{dz} \times \frac{dz}{dx}$$

So

$$\frac{dy}{dx} = \cos z \times 5 \qquad \text{since} \qquad \frac{dz}{dx} = 5$$

Then, finally

$$\frac{dy}{dx} = 5\cos z = 5\cos(5x + 3)$$

The chain rule: if $y(z)$ is a function of z and $z(x)$ is a function of x, then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}z} \times \frac{\mathrm{d}z}{\mathrm{d}x}$$

Example

Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ when $y = \mathrm{e}^{(x^2)}$.

Solution

x^2 is a function, so $\mathrm{e}^{(x^2)}$ is a function of a function. If we let $z = x^2$, then $y = \mathrm{e}^z$. Then

$$\frac{\mathrm{d}z}{\mathrm{d}x} = 2x \qquad \text{and} \qquad \frac{\mathrm{d}y}{\mathrm{d}z} = \mathrm{e}^z$$

so that, using the chain rule,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}z} \times \frac{\mathrm{d}z}{\mathrm{d}x} = \mathrm{e}^z \times 2x = 2x\mathrm{e}^{(x^2)}$$

Example

If $y = \sin^3 x$ find $\frac{\mathrm{d}y}{\mathrm{d}x}$.

Solution

First of all note that $\sin^3 x$ means $(\sin x)^3$. Therefore y can be written $y = (\sin x)^3$, so that this is a function of a function.

If we let $z = \sin x$ then $y = z^3$. It follows that

$$\frac{\mathrm{d}z}{\mathrm{d}x} = \cos x \qquad \text{and} \qquad \frac{\mathrm{d}y}{\mathrm{d}z} = 3z^2$$

Then, using the chain rule,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}z} \times \frac{\mathrm{d}z}{\mathrm{d}x} = 3z^2 \times \cos x = 3\sin^2 x \cos x$$

Exercises

In each case find $\frac{\mathrm{d}y}{\mathrm{d}x}$.

1. $y = \sin(x^2)$
2. $y = (\sin x)^2$
3. $y = \log_e(x^2 + 1)$
4. $y = (2x + 7)^8$
5. $y = \mathrm{e}^{2x-3}$

Answers

1. $2x\ \cos(x^2)$. 2. $2\sin x \cos x$. 3. $\frac{2x}{x^2+1}$. 4. $16(2x + 7)^7$. 5. $2\mathrm{e}^{2x-3}$.

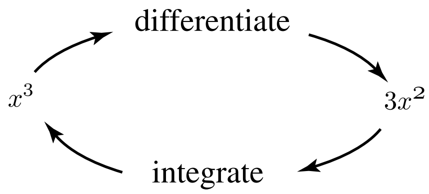
Integration as the reverse of differentiation

Introduction

Integration can be introduced in several different ways. One way is to think of it as differentiation in reverse. This approach is described in this leaflet.

1. Differentiation in reverse

Suppose we differentiate the function $y = x^3$. We obtain $\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2$. Integration reverses this process and we say that the integral of $3x^2$ is x^3. Pictorially we can think of this as follows:



The situation is just a little more complicated because there are lots of functions we can differentiate to give $3x^2$. Here are some of them:

$$x^3 + 14, \qquad x^3 + 7, \qquad x^3 - 0.25, \qquad x^3 - \frac{1}{2}$$

Each of these functions has the same derivative, $3x^2$, because when we differentiate the constant term we obtain zero. Consequently, when we try to reverse the process, we have no idea what the original constant term might have been. Because of this we include in our answer an unknown constant, c say, called the **constant of integration.** We state that the integral of $3x^2$ is $x^3 + c$.

The symbol for integration is $\int$, known as an **integral sign**. Formally we write

$$\int 3x^2\,\mathrm{d}x = x^3 + c$$

Along with the integral sign there is a term '$\mathrm{d}x$', which must always be written, and which indicates the name of the variable involved, in this case x. Technically, integrals of this sort are called **indefinite integrals**, to distinguish them from definite integrals which are dealt with in a subsequent leaflet. When asked to find an indefinite integral your answer should always contain a constant of integration.

Common integrals are usually found in a 'Table of Integrals' such as that shown here. A more complete table is available in leaflet *8.7 Table of integrals*.

Table of integrals

Function $f(x)$	Indefinite integral $\int f(x)\mathrm{d}x$
constant, k	$kx + c$
x	$\frac{x^2}{2} + c$
x^2	$\frac{x^3}{3} + c$
x^n	$\frac{x^{n+1}}{n+1} + c \quad n \neq -1$
$\sin x$	$-\cos x + c$
$\cos x$	$\sin x + c$
$\sin kx$	$\frac{-\cos kx}{k} + c$
$\cos kx$	$\frac{\sin kx}{k} + c$
$\tan kx$	$\frac{1}{k} \ln \lvert \sec kx \rvert$+c
e^x	$\mathrm{e}^x + c$
e^{-x}	$-\mathrm{e}^{-x} + c$
e^{kx}	$\frac{\mathrm{e}^{kx}}{k} + c$
$x^{-1} = \frac{1}{x}$	$\ln \lvert x \rvert + c$

When dealing with the trigonometric functions the variable x must always be measured in radians.

Example

Use the table above to find a) $\int x^8 \mathrm{d}x$, b) $\int x^{-4} \mathrm{d}x$.

Solution

From the table note that

$$\int x^n \mathrm{d}x = \frac{x^{n+1}}{n+1} + c$$

a) With $n = 8$ we find

$$\int x^8 \mathrm{d}x = \frac{x^{8+1}}{8+1} + c = \frac{x^9}{9} + c$$

b) With $n = -4$ we find

$$\int x^{-4} \mathrm{d}x = \frac{x^{-4+1}}{-4+1} + c = \frac{x^{-3}}{-3} + c$$

Note that the final answer can be written in a variety of equivalent ways, for example

$$-\frac{1}{3}x^{-3} + c, \qquad \text{or} \qquad -\frac{1}{3} \cdot \frac{1}{x^3} + c, \qquad \text{or} \qquad -\frac{1}{3x^3} + c$$

Exercises

1. Integrate each of the following functions:

a) x^9, b) $x^{1/2}$, c) x^{-3}, d) $\frac{1}{x^4}$, e) 4, f) $\sqrt{x}$, g) e^{4x}, h) 17, i) $\cos 5x$.

Answers

1. a) $\frac{x^{10}}{10} + c$, b) $\frac{2x^{3/2}}{3} + c$, c) $-\frac{1}{2}x^{-2} + c$, d) $-\frac{1}{3}x^{-3} + c$,

e) $4x + c$, f) same as b), g) $\frac{\mathrm{e}^{4x}}{4} + c$, h) $17x + c$, i) $\frac{\sin 5x}{5} + c$.

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8.7

Table of integrals

Engineers usually refer to a table of integrals when performing calculations involving integration. This leaflet provides such a table. Sometimes restrictions need to be placed on the values of some of the variables. These restrictions are shown in the third column.

1. A table of integrals

$f(x)$	$\int f(x)\,\mathrm{d}x$	
k, any constant	$kx + c$	
x	$\frac{x^2}{2} + c$	
x^2	$\frac{x^3}{3} + c$	
x^n	$\frac{x^{n+1}}{n+1} + c$	$n \neq -1$
$x^{-1} = \frac{1}{x}$	$\ln\|x\| + c$	
e^x	$\mathrm{e}^x + c$	
e^{kx}	$\frac{1}{k}\mathrm{e}^{kx} + c$	
$\cos x$	$\sin x + c$	
$\cos kx$	$\frac{1}{k}\sin kx + c$	
$\sin x$	$-\cos x + c$	
$\sin kx$	$-\frac{1}{k}\cos kx + c$	
$\tan x$	$\ln(\sec x) + c$	$-\frac{\pi}{2} < x < \frac{\pi}{2}$
$\sec x$	$\ln(\sec x + \tan x) + c$	$-\frac{\pi}{2} < x < \frac{\pi}{2}$
$\mathrm{cosec}\, x$	$\ln(\mathrm{cosec}\, x - \cot x) + c$	$0 < x < \pi$
$\cot x$	$\ln(\sin x) + c$	$0 < x < \pi$
$\cosh x$	$\sinh x + c$	
$\sinh x$	$\cosh x + c$	
$\tanh x$	$\ln\cosh x + c$	
$\coth x$	$\ln\sinh x + c$	$x > 0$
$\frac{1}{x^2+a^2}$	$\frac{1}{a}\tan^{-1}\frac{x}{a} + c$	$a > 0$
$\frac{1}{x^2-a^2}$	$\frac{1}{2a}\ln\frac{x-a}{x+a} + c$	$\|x\| > a > 0$
$\frac{1}{a^2-x^2}$	$\frac{1}{2a}\ln\frac{a+x}{a-x} + c$	$\|x\| < a$
$\frac{1}{\sqrt{x^2+a^2}}$	$\sinh^{-1}\frac{x}{a} + c$	$a > 0$
$\frac{1}{\sqrt{x^2-a^2}}$	$\cosh^{-1}\frac{x}{a} + c$	$x \geqslant a > 0$
$\frac{1}{\sqrt{x^2+k}}$	$\ln(x + \sqrt{x^2+k}) + c$	
$\frac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1}\frac{x}{a} + c$	$-a \leqslant x \leqslant a$

Exercises

1. In each case, use the Table of Integrals to integrate the given function with respect to x.

a) x

b) x^6

c) x^{-2}

d) x^{-3}

e) x^{-1} (be careful!)

f) $x^{1/2}$

g) $x^{-1/2}$

h) e^{3x}

i) e^{7x}

j) e^{-2x}

k) $\mathrm{e}^{0.5x}$

l) e^{x}

m) e^{-x}

n) $\cos x$

o) $\sin x$

p) $\sin 3x$

q) $\cos 2x$

r) 5

2. You should be able to use the table when variables other than x are involved. Use the table to integrate each of the following functions with respect to t.

a) e^t, b) e^{5t}, c) t^7, d) $\sqrt{t}$, e) $\cos 5t$, f) e^{-t}.

Answers

1. a) $\frac{x^2}{2}+c$, b) $\frac{x^7}{7}+c$, c) $\frac{x^{-1}}{-1}+c=-x^{-1}+c$, or $-\frac{1}{x}+c$, d) $\frac{x^{-2}}{-2}+c=-\frac{1}{2}x^{-2}+c$, or $-\frac{1}{2x^2}+c$, e) $\ln|x|+c$, f) $\frac{x^{3/2}}{3/2}+c=\frac{2}{3}x^{3/2}+c$, g) $\frac{x^{1/2}}{1/2}+c=2x^{1/2}+c$, h) $\frac{1}{3}\mathrm{e}^{3x}+c$, i) $\frac{1}{7}\mathrm{e}^{7x}+c$, j) $-\frac{1}{2}\mathrm{e}^{-2x}+c$, k) $2\mathrm{e}^{0.5x}+c$, l) e^x+c, m) $-\mathrm{e}^{-x}+c$, n) $\sin x+c$, o) $-\cos x+c$, p) $-\frac{1}{3}\cos 3x+c$, q) $\frac{1}{2}\sin 2x+c$, r) $5x+c$.

2. a) e^t+c, b) $\frac{\mathrm{e}^{5t}}{5}+c$, c) $\frac{t^8}{8}+c$, d) $\frac{2t^{3/2}}{3}+c$, e) $\frac{\sin 5t}{5}+c$, f) $-\mathrm{e}^{-t}+c$.

Linearity rules of integration

Introduction

To enable us to find integrals of a wider range of functions than those normally given in a Table of Integrals we can make use of two rules known as **linearity rules**.

1. The integral of a constant multiple of a function

A constant factor in an integral can be moved outside the integral sign in the following way.

$$\int k\, f(x)\, \mathrm{d}x = k \int f(x) \mathrm{d}x$$

This is only possible when k is a constant, and it multiplies some function of x.

Example
Find $\int 11x^2\, \mathrm{d}x$.

Solution
We are integrating a multiple of x^2. The constant factor,11, can be moved outside the integral sign.

$$\int 11x^2\, \mathrm{d}x = 11 \int x^2\, \mathrm{d}x = 11\left(\frac{x^3}{3} + c\right) = \frac{11x^3}{3} + 11c$$

where c is the constant of integration. Because $11c$ is a constant we would normally write the answer in the form $\frac{11x^3}{3} + K$ where K is another constant.

Example
Find $\int -5\cos x\, \mathrm{d}x$.

Solution
We are integrating a multiple of $\cos x$. The constant factor, -5, can be moved outside the integral sign.

$$\int -5\cos x\, \mathrm{d}x = -5\int \cos x\, \mathrm{d}x = -5\,(\sin x + c) = -5\sin x + K$$

where K is a constant.

2. The integral of the sum or difference of two functions

When we wish to integrate the sum or difference of two functions, we integrate each term separately as follows:

$$\int f(x) + g(x) \quad \mathrm{d}x = \int f(x)\,\mathrm{d}x + \int g(x)\,\mathrm{d}x$$

$$\int f(x) - g(x) \quad \mathrm{d}x = \int f(x)\,\mathrm{d}x - \int g(x)\,\mathrm{d}x$$

Example
Find $\int (x^3 + \sin x)\mathrm{d}x$.

Solution

$$\int (x^3 + \sin x)\mathrm{d}x = \int x^3\,\mathrm{d}x + \int \sin x\,\mathrm{d}x = \frac{x^4}{4} - \cos x + c$$

Note that only a single constant of integration is needed.

Example
Find $\int \mathrm{e}^{3x} - x^7\,\mathrm{d}x$.

Solution

$$\int \mathrm{e}^{3x} - x^7\,\mathrm{d}x = \int \mathrm{e}^{3x}\,\mathrm{d}x \quad - \quad \int x^7\,\mathrm{d}x = \frac{\mathrm{e}^{3x}}{3} - \frac{x^8}{8} + c$$

Exercises
1. Find a) $\int 8x^5 + 3x^2\,\mathrm{d}x$, b) $\int \frac{2}{3}x\,\mathrm{d}x$.
2. Find $\int 3\cos x + 7x^3\,\mathrm{d}x$.
3. Find $\int 7x^{-2}\mathrm{d}x$.
4. Find $\int \frac{5}{x}\mathrm{d}x$.
5. Find $\int \frac{x+\cos 2x}{3}\mathrm{d}x$.
6. Find $\int 5\mathrm{e}^{4x}\mathrm{d}x$.
7. Find $\int \frac{\mathrm{e}^x - \mathrm{e}^{-x}}{2}\mathrm{d}x$.

Answers
1. a) $\frac{4x^6}{3} + x^3 + c$, b) $\frac{1}{3}x^2 + c$. 2. $3\sin x + \frac{7x^4}{4} + c$. 3. $-\frac{7}{x} + c$. 4. $5\log_e|x| + c$.
5. $\frac{x^2}{6} + \frac{\sin 2x}{6} + c$. 6. $\frac{5\mathrm{e}^{4x}}{4} + c$. 7. $\frac{\mathrm{e}^x + \mathrm{e}^{-x}}{2} + c$.

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8.9

Evaluating definite integrals

Introduction

Definite integrals can be recognised by numbers written to the upper and lower right of the integral sign. This leaflet explains how to evaluate definite integrals.

1. Definite integrals

The quantity

$$\int_a^b f(x)\,\mathrm{d}x$$

is called the **definite integral** of $f(x)$ from a to b. The numbers a and b are known as the **lower** and **upper limits** of the integral. To see how to evaluate a definite integral consider the following example.

Example

Find $\int_1^4 x^2 \mathrm{d}x$.

Solution

First of all the integration of x^2 is performed in the normal way. However, to show we are dealing with a definite integral, the result is usually enclosed in square brackets and the limits of integration are written on the right bracket:

$$\int_1^4 x^2\,\mathrm{d}x = \left[\frac{x^3}{3} + c\right]_1^4$$

Then, the quantity in the square brackets is evaluated, first by letting x take the value of the upper limit, then by letting x take the value of the lower limit. The difference between these two results gives the value of the definite integral:

$$\begin{aligned}
\left[\frac{x^3}{3} + c\right]_1^4 &= \text{(evaluate at upper limit)} - \text{(evaluate at lower limit)} \\
&= \left(\frac{4^3}{3} + c\right) - \left(\frac{1^3}{3} + c\right) \\
&= \frac{64}{3} - \frac{1}{3} \\
&= 21
\end{aligned}$$

Note that the constants of integration cancel out. This will always happen, and so in future we can ignore them when we are evaluating definite integrals.

Example
Find $\int_{-2}^{3} x^3 \mathrm{d}x$.

Solution

$$\begin{aligned}
\int_{-2}^{3} x^3 \mathrm{d}x &= \left[\frac{x^4}{4}\right]_{-2}^{3} \\
&= \left(\frac{(3)^4}{4}\right) - \left(\frac{(-2)^4}{4}\right) \\
&= \frac{81}{4} - \frac{16}{4} \\
&= \frac{65}{4} \\
&= 16.25
\end{aligned}$$

Example
Find $\int_{0}^{\pi/2} \cos x \, \mathrm{d}x$.

Solution

$$\begin{aligned}
\int_{0}^{\pi/2} \cos x \, \mathrm{d}x &= [\sin x]_0^{\pi/2} \\
&= \sin\left(\frac{\pi}{2}\right) - \sin 0 \\
&= 1 - 0 \\
&= 1
\end{aligned}$$

Exercises

1. Evaluate

a) $\int_0^1 x^2 \mathrm{d}x$, b) $\int_2^3 \frac{1}{x^2} \mathrm{d}x$, c) $\int_1^2 x^2 \mathrm{d}x$, d) $\int_0^4 x^3 \mathrm{d}x$, e) $\int_{-1}^1 x^3 \mathrm{d}x$.

2. Evaluate $\int_3^4 x + 7x^2 \, \mathrm{d}x$.

3. Evaluate a) $\int_0^1 \mathrm{e}^{2x} \mathrm{d}x$, b) $\int_0^2 \mathrm{e}^{-x} \mathrm{d}x$, c) $\int_{-1}^1 x^2 \mathrm{d}x$, d) $\int_{-1}^1 5x^3 \mathrm{d}x$.

4. Find $\int_{0}^{\pi/2} \sin x \mathrm{d}x$.

Answers

1. a) $\frac{1}{3}$, b) $\frac{1}{6}$, c) $\frac{7}{3}$, d) 64, e) 0.

2. 89.833 (3dp).

3. a) $\frac{\mathrm{e}^2}{2} - \frac{1}{2} = 3.195$ (3dp), b) $1 - \mathrm{e}^{-2} = 0.865$ (3dp), c) $\frac{2}{3}$, d) 0.

4. 1.

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8.10

Integration by parts

Introduction

The technique known as **integration by parts** is used to integrate a product of two functions, for example

$$\int e^{2x} \sin 3x \, dx \qquad \text{and} \qquad \int_0^1 x^3 e^{-2x} \, dx$$

This leaflet explains how to apply this technique.

1. The integration by parts formula

We need to make use of the integration by parts formula which states:

$$\int u \left(\frac{dv}{dx}\right) dx = uv - \int v \left(\frac{du}{dx}\right) dx$$

Note that the formula replaces one integral, the one on the left, with a different integral, that on the right. The intention is that the latter is simpler to evaluate. Note also that to apply the formula we must let one function in the product equal u. We must be able to differentiate this function to find $\frac{du}{dx}$. We let the other function in the product equal $\frac{dv}{dx}$. We must be able to integrate this function, to find v. Consider the following example:

Example

Find $\int 3x \sin x \, dx$.

Solution

Compare the required integral with the formula for integration by parts: we see that it makes sense to choose

$$u = 3x \qquad \text{and} \qquad \frac{dv}{dx} = \sin x$$

It follows that

$$\frac{du}{dx} = 3 \qquad \text{and} \qquad v = \int \sin x \, dx = -\cos x$$

(When integrating $\frac{dv}{dx}$ to find v there is no need to include a constant of integration. When you become confident with the method, you may like to think about why this is the case.) Applying the formula we obtain

$$\begin{aligned}
\int 3x\sin x\,\mathrm{d}x &= uv - \int v\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)\mathrm{d}x \\
&= 3x(-\cos x) - \int(-\cos x).(3)\,\mathrm{d}x \\
&= -3x\cos x + 3\int \cos x\,\mathrm{d}x \\
&= -3x\cos x + 3\sin x + c
\end{aligned}$$

2. Dealing with definite integrals

When dealing with definite integrals (those with limits of integration) the corresponding formula is

$$\int_a^b u\left(\frac{\mathrm{d}v}{\mathrm{d}x}\right)\mathrm{d}x = [uv]_a^b - \int_a^b v\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)\mathrm{d}x$$

Example

Find $\displaystyle\int_0^2 x\mathrm{e}^x\mathrm{d}x$.

Solution

We let $u = x$ and $\dfrac{\mathrm{d}v}{\mathrm{d}x} = \mathrm{e}^x$. Then $\dfrac{\mathrm{d}u}{\mathrm{d}x} = 1$ and $v = \mathrm{e}^x$. Using the formula for integration by parts we obtain

$$\begin{aligned}
\int_0^2 x\mathrm{e}^x\mathrm{d}x &= [x\mathrm{e}^x]_0^2 - \int_0^2 \mathrm{e}^x.1\mathrm{d}x \\
&= (2\mathrm{e}^2) - (0\mathrm{e}^0) - [\mathrm{e}^x]_0^2 \\
&= 2\mathrm{e}^2 - [\mathrm{e}^2 - 1] \\
&= \mathrm{e}^2 + 1 \qquad \text{(or 8.389 to 3dp)}
\end{aligned}$$

Exercises

1. Find a) $\int x\sin(2x)\mathrm{d}x$, b) $\int t\mathrm{e}^{3t}\mathrm{d}t$, c) $\int x\cos x\,\mathrm{d}x$.

2. Evaluate the following definite integrals:

a) $\int_0^1 x\cos 2x\,\mathrm{d}x$, b) $\int_0^{\pi/2} x\sin 2x\,\mathrm{d}x$, c) $\int_{-1}^1 t\mathrm{e}^{2t}\mathrm{d}t$.

(Remember to set your calculator to radian mode for evaluating the trigonometric functions.)

3. Find $\int_0^2 x^2\mathrm{e}^x\mathrm{d}x$. (You will need to apply the integration by parts formula twice.)

Answers

1. a) $\frac{\sin 2x}{4} - \frac{x\cos 2x}{2} + c$, b) $\mathrm{e}^{3t}(\frac{t}{3} - \frac{1}{9}) + c$, c) $\cos x + x\sin x + c$.

2. a) 0.1006, b) 0.7854, c) 1.9488.

3. 12.778 (3dp).

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8.11

Integration by substitution

Introduction

This technique involves making a substitution in order to simplify an integral before evaluating it. We let a new variable, u say, equal a more complicated part of the function we are trying to integrate. The choice of which substitution to make often relies upon experience: don't worry if at first you cannot see an appropriate substitution. This skill develops with practice.

1. Making a substitution

Example
Find $\int (3x+5)^6 \mathrm{d}x$.

Solution
First look at the function we are trying to integrate: $(3x+5)^6$. Suppose we introduce a new variable, u, such that $u = 3x+5$. Doing this means that the function we must integrate becomes u^6. This certainly looks a much simpler function to integrate than $(3x+5)^6$. There is a slight complication however. The new function of u must be integrated with respect to u and not with respect to x. This means that we must take care of the term $\mathrm{d}x$ correctly. From the substitution

$$u = 3x + 5$$

note, by differentiation, that

$$\frac{\mathrm{d}u}{\mathrm{d}x} = 3$$

It follows that we can write

$$\mathrm{d}x = \frac{\mathrm{d}u}{3}$$

The required integral then becomes

$$\int (3x+5)^6 \mathrm{d}x = \int u^6 \frac{\mathrm{d}u}{3}$$

The factor of $\frac{1}{3}$, being a constant, means that we can write

$$\begin{aligned} \int (3x+5)^6 \mathrm{d}x &= \frac{1}{3}\int u^6 \mathrm{d}u \\ &= \frac{1}{3}\frac{u^7}{7} + c \\ &= \frac{u^7}{21} + c \end{aligned}$$

To finish off we rewrite this answer in terms of the original variable, x, and replace u by $3x+5$:

$$\int (3x+5)^6 \mathrm{d}x = \frac{(3x+5)^7}{21} + c$$

2. Substitution and definite integrals

If you are dealing with definite integrals (ones with limits of integration) you must be particularly careful with the way you handle the limits. Consider the following example.

Example
Find $\int_2^3 t\sin(t^2)\mathrm{d}t$ by making the substitution $u = t^2$.

Solution
Note that if $u = t^2$ then $\frac{\mathrm{d}u}{\mathrm{d}t} = 2t$ so that $\mathrm{d}t = \frac{\mathrm{d}u}{2t}$. We find

$$\begin{aligned}\int_{t=2}^{t=3} t\sin(t^2)\mathrm{d}t &= \int_{t=2}^{t=3} t\sin u\,\frac{\mathrm{d}u}{2t} \\ &= \frac{1}{2}\int_{t=2}^{t=3} \sin u\,\mathrm{d}u\end{aligned}$$

An important point to note is that the original limits of integration are limits on the variable t, not u. To emphasise this they have been written explicitly as $t = 2$ and $t = 3$. When we integrate with respect to the variable u, the limits must be written in terms of u too. From the substitution $u = t^2$, note that

$$\text{when } t = 2, u = 4 \quad \text{and when} \quad t = 3, u = 9$$

so the integral becomes

$$\begin{aligned}\frac{1}{2}\int_{u=4}^{u=9} \sin u\,\mathrm{d}u &= \frac{1}{2}\left[-\cos u\right]_4^9 \\ &= \frac{1}{2}(-\cos 9 + \cos 4) \\ &= 0.129\end{aligned}$$

Exercises
1. Use a substitution to find

a) $\int (4x+1)^7 \mathrm{d}x$,   b) $\int_1^2 (2x+3)^7 \mathrm{d}x$,   c) $\int t^2\sin(t^3+1)\mathrm{d}t$,   d) $\int_0^1 3t^2 \mathrm{e}^{t^3}\mathrm{d}t$.

2. Make a substitution to find the following integrals. Can you deduce a rule for integrating functions of the form $\frac{f'(x)}{f(x)}$?

a) $\int \frac{1}{x+1}\mathrm{d}x$,   b) $\int \frac{2x}{x^2+7}\mathrm{d}x$,   c) $\int \frac{3x^2}{x^3+17}\mathrm{d}x$.

Answers
1. a) $\frac{(4x+1)^8}{32} + c$,   b) 3.3588×10^5,   c) $-\frac{\cos(t^3+1)}{3} + c$,   d) 1.7183.

2. a) $\ln(x+1) + c$,   b) $\ln(x^2+7) + c$,   c) $\ln(x^3+17) + c$.

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8.12

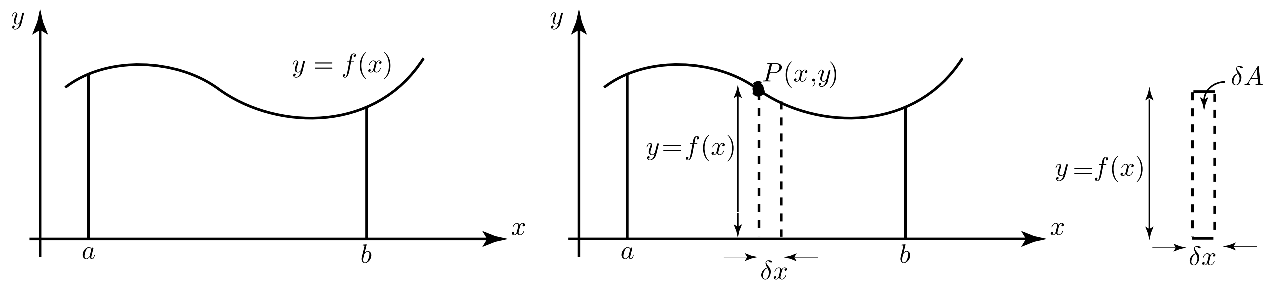
Integration as summation

Introduction

In this leaflet we explain integration as an infinite sum.

1. Integration as summation

The figure below on the left shows an area bounded by the x axis, the lines $x = a$ and $x = b$, and the curve $y = f(x)$. Note that the area lies entirely above the x axis.



There are several ways in which this area can be estimated. Suppose we split the area into thin vertical strips, like the one shown, and treat each strip as being approximately rectangular. The sum of the areas of the rectangular strips then gives an approximate value for the area under the curve. The thinner the strips, the better will be the approximation. A typical strip is shown drawn from the point $P(x, y)$. The width of the strip is labelled δx. We label it like this because the symbol δ is used to indicate a small increase in the variable being considered, in this case x. The height of the strip is equal to the y value on the curve at point P, that is $f(x)$. So the area of the strip shown is approximately $f(x)\,\delta x$. Suppose we let the area of this small strip be δA. We use the delta notation again, because this strip makes a small contribution, δA, to the total area, A, under the curve. Then

$$\delta A \approx f(x)\,\delta x$$

Now if we add up the areas of all such thin strips from a to b, which we denote by $\sum_{x=a}^{b} \delta A$, we obtain the total area under the curve.

$$\text{total area} = \sum_{x=a}^{b} \delta A \approx \sum_{x=a}^{b} f(x)\,\delta x$$

To make this approximation more accurate we must let the thickness of each strip become very small indeed, that is, we let $\delta x \to 0$, giving

$$\text{total area} = \lim_{\delta x \to 0} \sum_{x=a}^{b} f(x)\delta x$$

The notation $\lim_{\delta x \to 0}$ means that we consider what happens to the expression following it as δx gets smaller and smaller. This is known as the **limit of a sum**. If this limit exists we write it formally as

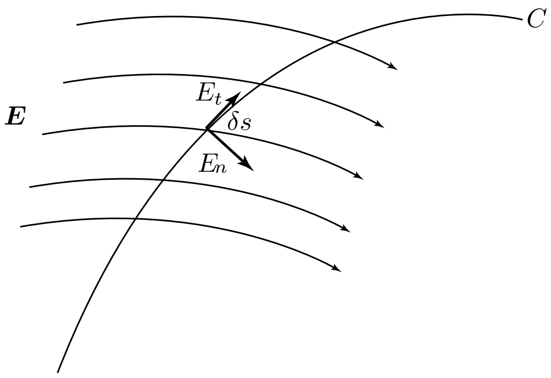
$$\int_a^b f(x)\,\mathrm{d}x$$

thus defining a definite integral as the limit of a sum. Thus we have the important result that

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{\delta x \to 0} \sum_{x=a}^{b} f(x)\,\delta x$$

Integration can therefore be regarded as a process of adding up, that is as a **summation**. Whenever we wish to find areas under curves, volumes etc., we can do this by finding the area or volume of a small portion, and then summing over the whole region of interest. The calculation can then be performed using the technique of definite integration.

Example



Suppose a unit charge moves along a curve C in an electric field $\mathbf{E}$. At any point on the curve the electric field vector can be resolved into two perpendicular components, E_t say, along the curve, and E_n perpendicular, or normal, to the curve. In moving the charge a small distance δs along the curve the electric field does work equal to $E_t \delta s$, because only the tangential component does work. To find the total work done as the charge moves along the length of the curve we must sum all such small contributions, i.e.

$$\text{total work done } = \sum E_t\,\delta s, \qquad \text{in the limit as } \delta s \to 0$$

that is

$$\text{total work done } = \lim_{\delta s \to 0} \sum E_t\,\delta s$$

which defines the integral $\int_C E_t \mathrm{d}s$. The symbol $\int_C$ tells us to sum the contributions along the curve C. This is an example of a **line integral** because we integrate along the line (curve) C.

Exercises

1. Write down, but do not calculate, the integral which is defined by the limit as $\delta x \to 0$, of the following sums.

a) $\sum_{x=3}^{x=5} 7x^2\,\delta x$, b) $\sum_{x=1}^{x=7} \frac{4}{3}\pi x^3\,\delta x$.

Answers

1. a) $\int_3^5 7x^2 \mathrm{d}x$, b) $\int_1^7 \frac{4}{3}\pi x^3 \mathrm{d}x$.

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